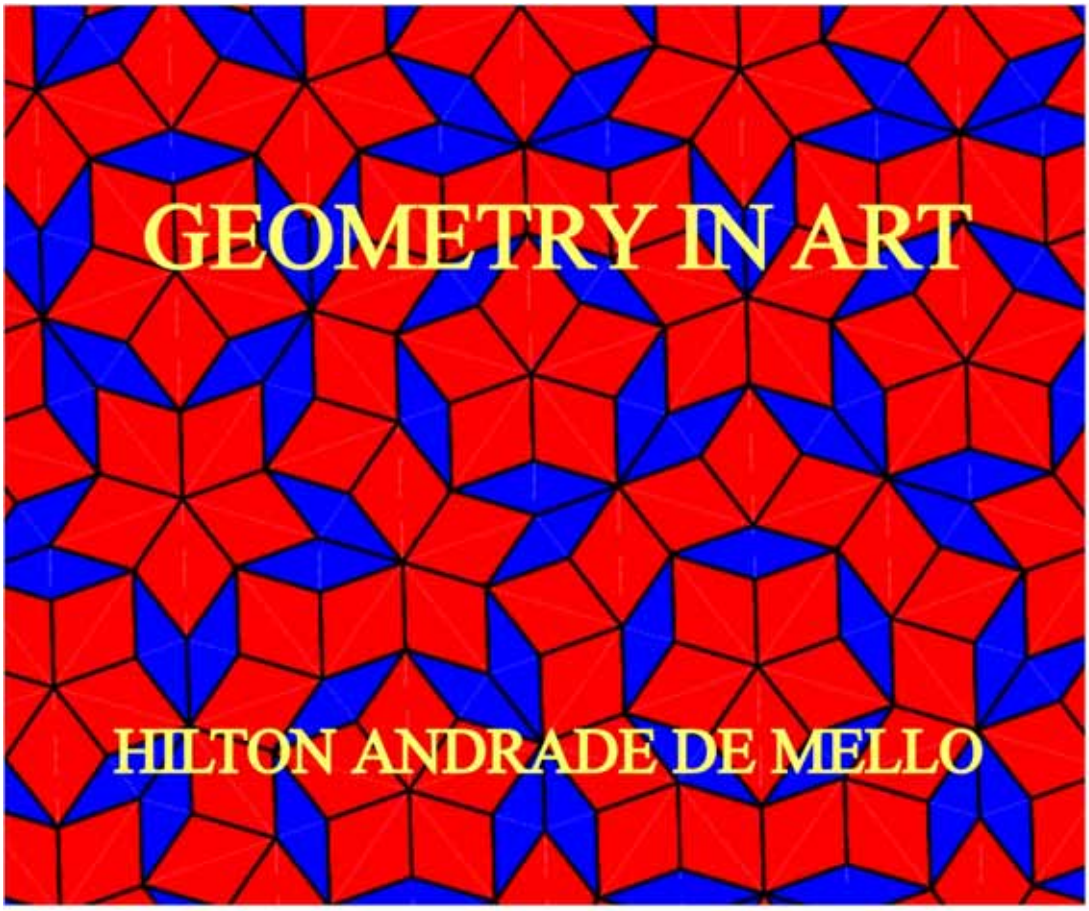
GEOMETRY IN ART
HILTON ANDRADE DE MELLO

# GEOMETRY IN ART
# Hilton Andrade de Mello

Translated from the Portuguese, “Geometria nas Artes”, by
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To Leonardo, Rafael, Ana Clara and Dominique, whose joy and creativity inspired me throughout this work.

And to the memory of Paula, that enlightened and extraordinary spirit that guided our paths with joy.

HAM (Hamello)

Do not let enter anyone who has no knowledge of geometry

In 387 BC Plato started his academy of philosophy in Athens, which existed until closed by Emperor Justinian in 529 AD. The words above were at the entrance to the academy.

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Obviously, no justice is done to the importance of their work by viewing only one or two of their artwork, as they are, without any doubt, the exponents of the art in Brazil.

I am indebted to João Carlos Galvão, for making it possible for me to establish the contacts I have in the art world.

From abroad, I acknowledge the significant contributions I received from Dick Termes, Barry Stevens, George Hart, Philippe Hurbain, Russel Towle, Steve Frisby, and Dar Freeland.

My thanks to all, hoping that I have contributed to the dissemination of geometric art.

HAM (Hamello)

## About the author

**Hilton Andrade de Mello** is a graduate of the Federal University of Rio de Janeiro, Brazil, from which he received his electrical/electronics engineering degree, and his Nuclear Engineering degree, both in 1962. He also holds graduate degrees from Stanford University, U.S.A., where he majored in Electrical Engineering, concentrating in electronics. His scientific career has been, since graduation, at the National Nuclear Energy Commission ("Comissão Nacional de Energia Nuclear"), in Brazil, and he is the author and co-author of several electronics textbooks (listed in www.hamello.com.br). At present he is devoted to the Plastic Arts, and has many paintings to his credit.

# Introduction

Having participated in many educational activities, I had the opportunity to observe that newcomers to plastic arts, especially those interested in geometric abstraction, are generally unfamiliar with concepts related to geometric forms. There are a number of technical books on the subject matter but they are generally not written for those who are plastic artists. Such books generally do not show the relationship between geometric forms and the different nuances of the art.

It is my intention in this book to "cruise the geometric forms world" in order to familiarize the reader with such an area of knowledge. That area, which is based on the work of the Greek mathematician Euclid, gave birth to the so-called *Euclidean Geometry*.

Our journey starts in Chapter 1 with Manet and the impressionists around the second half of the 19th century. Their objective was not to paint a true copy of either an object or a landscape, but to stress the sensations of motion and light in their work. We then proceed to Russia, at the time of Lenin, where we notice the importance of the works of Wassili Kandinsky and Kasimir Malevich. Passing briefly through The Netherlands we get acquainted with the work of Piet Mondrian, the "Stijl group", and with the manifest of the *Concrete Art* of Theo Van Doesburg. We also get acquainted, in Paris, with the "Circle and Square" movement lead by the Belgian art critic and artist Michel Seuphor, and by the Uruguayan painter Joaquim Torres Garcia.

We then leave Europe and travel to Brazil, in time for the opening of the São Paulo Museum of Modern Art. While in São Paulo we will also visit the 1st São Paulo Art Biennial, where the work of several Brazilian artists were exhibited, and witness the award ceremonies for Max Bill (celebrating one of his innovative sculptures) and for Ivan Serpa (celebrating his work entitled "Forms"), and see the intriguing kinechromatic work of Abraham Palatnik.

While in São Paulo we will also visit Samson Flexor's "Atelier Abstração", and meet the Noigandres and Ruptura groups. Traveling to Rio de Janeiro we will meet the "Grupo Frente" and witness the birth of the "Manifesto da Arte Neo-Concreta".

We end this first leg of our journey by getting acquainted with several works by other Brazilian artists who were involved in the initial phase of the concrete art in Brazil.

In Chapter 2, the abstract concepts involving point, straight line, and plane, and geometric forms such as polygons, circles, stars, rosaceae, and spirals are presented. The works of several artists involving such forms are presented in that chapter.

In Chapter 3 the most important types of polyhedrons are presented, with especial attention given to the so-called Platonic Solids, known as such because they were investigated by the philosopher Plato. Examples of their use in the arts are presented.

Surfaces of revolution are presented in Chapter 4. They include spheres, cylinders, cones, conic surfaces and their associated solids, like the ellipsoid, the paraboloid, and the hyperboloid. Helices and helicoids are also presented. That chapter ends with a presentation of important works by Ascânio MMM.

In Chapter 5 it is shown how a planar surface, such as a canvas or a panel, can be completely filled leaving no void space on it. Such is the case of tessellations (also known as tilings), an example of which is a patchwork quilt.

In Chapter 6 the concept of linear perspective is presented and it is shown how such a technique enriched important works during the Renaissance. The basic concept of perspective is expanded in the creative spheres of Dick Termes, which are presented in that chapter.

Chapter 7 deals with the *Fibonacci sequence* and the *golden ratio*. Several artists may have used such concepts in their work, and examples by Da Vinci, including the famous Vitruvian Man, based on the studies of the roman architect Marcus Vitruvius Pollio, are presented. The golden ratio apparently appears in several species in nature, and that is also illustrated in that chapter.

Chapter 8 deals with the concepts of symmetry and of specular image, and of how the interpretation of an image can change when the image is seen through its specular form.

Our journey is now near its end and, at this point, it is shown in Chapter 9 how geometric forms and symbolisms are related to each other. For this, several studies are presented in that chapter. These include the *Vesica Piscis*, from which the symbol of ancient Christianity originates, the Pentagrams and the Hexagrams, the Labyrinths, and the Mandalas, with their spiritual association.

The journey then ends with a short introduction to informatics in Chapter 10, concentrating on information search on the Internet. Several computer programs one can use to learn more about geometric forms are listed. A number of references that were consulted by this author are listed at the end of the book.

It should be noted that this is not a book about art history, and that it does not deal with detailed study of any artistic movement or any artist. It only presents some of the work that involves geometry.

HAM (Hamello)

# 1

## Abstraction and Geometry

Figure 1.1 Untitled - Hamello

For several centuries, art was seen as a representation of the real world, be them objects, forms in nature, or people. During that time many artists supported themselves by painting important nobility, especially those connected to the Royal House.

With the advent of the Renaissance, works by artists such as Leonardo Da Vinci and Michelangelo started to appear, emphasizing more and more the notion of perfection. Perspective also appeared at that time, which is a method to represent 3-dimensional space on a flat surface.

The first artistic attempts to depart from an exact reproduction of reality were introduced by the "impressionists", who at first shocked the French society with their boldness and characteristic manner of representing objects, people and scenery.

Perhaps Edouard Manet (1832-1883) was the first of such artists who, instead of painting exactly what was seen, tried to reinforce other details such as light and motion. Several other artists followed Manet's idea in stressing the effects of light, which requires the observation of the object to be painted during several times of the day. In addition, to capture the effect of light on a scene it was imperative to use rapid strokes on the canvas, mixing the primary colors on the canvas instead of preparing the desired color on a palette. However, such a technique was not adopted by all the artists in Manet's group.

Figure 1.2 shows a work by Claude Monet (1840-1926) depicting a port seen through the morning fog. Monet was one of the artists in Manet's group, and became one of the best-known impressionist artists.

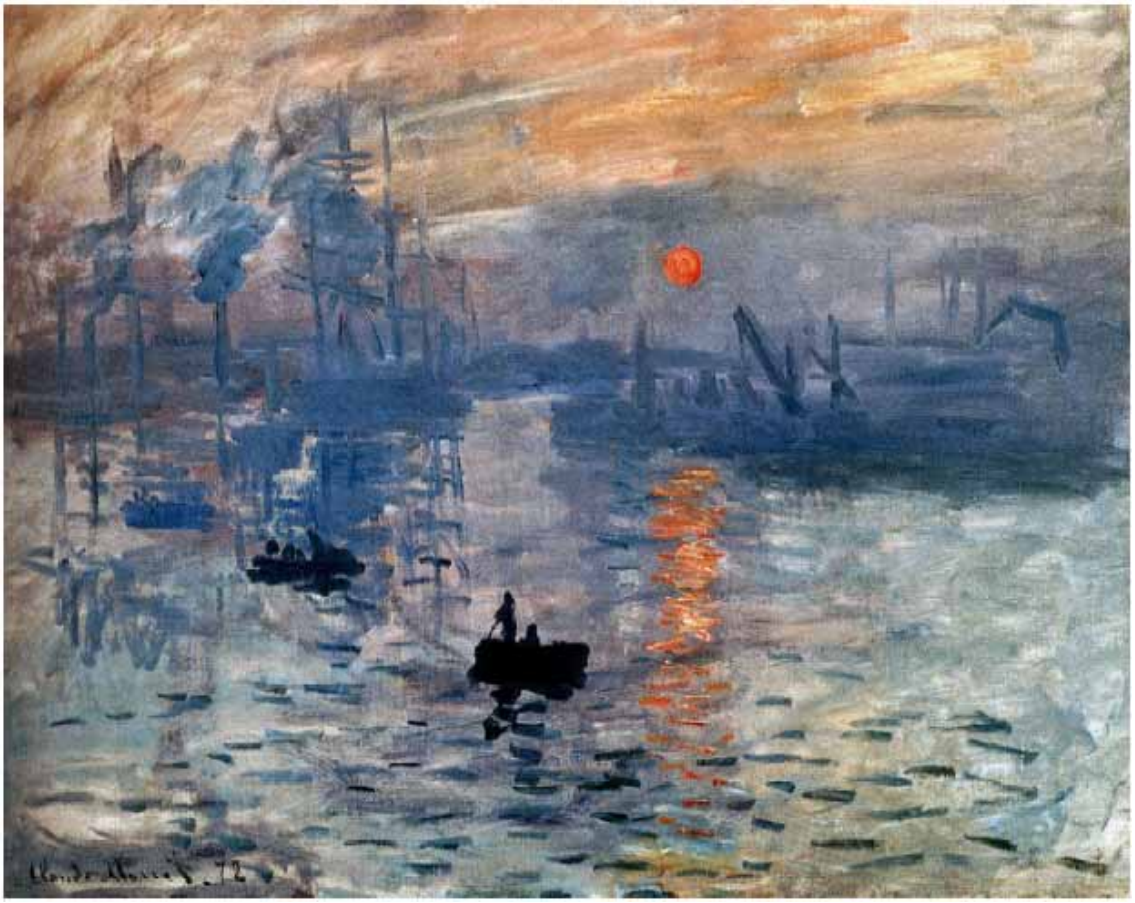

Figure 1.2 Sunrise - Claude Monet - Oil on canvas - Marmottan Monet Museum, Paris, France

Interestingly, when Monet's work shown in Figure 1.2 was shown in public in 1874, the catalog listed it as "Impression: the sunrise". This gave rise to the term impressionism, first used on April 15, 1874, by Louis Leroy in the Charivari magazine and meant as an insult to that group for presenting such an "exotic work"! The artists, however, did not take offense in that designation, and started to refer themselves as the

impressionists. As can be seen in Figure 1.2, real life objects such as boats, people, clouds, etc, still can be identified.

The departure from reality came later with the Russian painter Wassili Kandinsky (1866-1944) who created what became known as abstract art. In such an art form, he no longer sought to represent any existing object or form; instead, he used colors and lines to cause a sensorial impact at the viewer. This is illustrated in Figure 1.1.

At about the same time of Kandinsky, another Russian artist, Kasimir Malevich (1878-1935), introduced a more radical idea for abstract art. For him objects did not have any significance per se, and one should seek the "supremacy of pure sensation in creative art" as he himself described. That school of thought is known as Suprematism. Among his best-known works during that phase of his life are the "Black Square" (1915), shown in Figure 1.3, and the "White Square on a White Background" (1918).

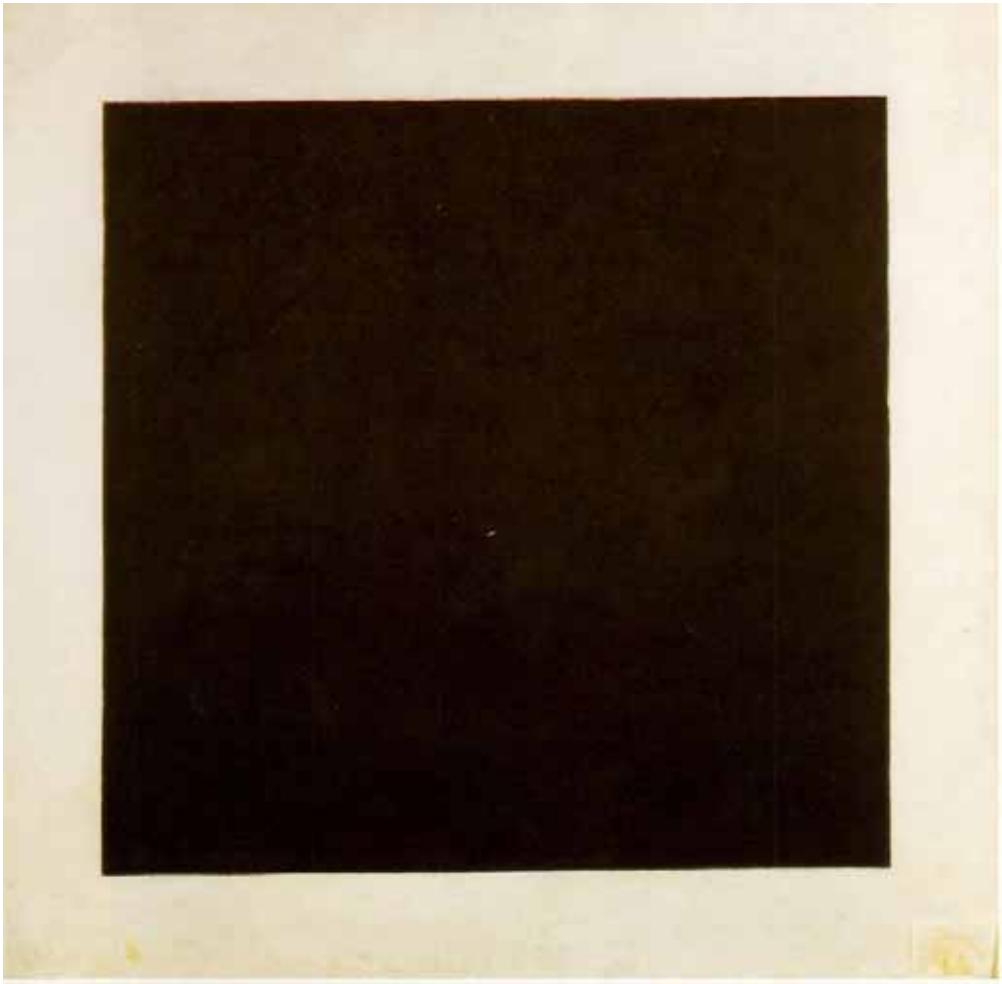

Figure 1.3 Black square - Malevich - Oil on canvas - Hermitage Museum, St. Petersburg, Russia

The Suprematism art, also known by some as geometric abstraction, caused serious problems for Malevich because for Lenin, who was in power at that time in Russia, art should be used for communication with the masses and not for advancing complex and innovative ideas to the people.

At this point one could ask why geometry was used to create abstract work?

When studying geometry we will see that it defines mathematical entities that are actually an idealization as one is really unable to paint a point, for example, since a point has no dimensions. The "point" that is drawn will have a finite dimension no matter how fine a pencil or a brush tip is. All geometric elements are actually abstract entities. Even in nature, when one looks at a flower and describes it as circular,

such a description is only an approximation because the ever-present irregularities prevent it from being a perfect circle.

As our senses are limited, it is understandable that one can look at an object and describe it is a square, a triangle, or a circle, for example.

Returning to the topic of abstract art, the use of geometric forms creating what is known as "geometric abstraction" is a perfectly valid process for an artist to "free himself/herself" from the real world.

Probably the Dutch painter Piet Mondrian (1872-1944) was the most influential geometric artist of his time. He adopted rigidly defined rules associated with geometric forms, such as using only black, white, and primary colors to obtain a great visual purity. Mondrian was part of a Dutch group called "De Stijl" (The Style), started by Theo Van Doesburg with several other Dutch artists. Van Doesburg also started in 1917 a periodical by the same name. The works of that group are important to the art world and we suggest the reader to consult an art history book for more details.

Like everything else that evolves with time, the ideas outlined above were not definitive, and Van Doesburg wrote, in 1930, the "Concrete Art Manifesto" in which he dismissed any symbolic or lyric connotation with art; in other words, a work of art embedding colors, lines, and other forms, should not have any special meaning but only its intrinsic value.

For Doesburg, the terms ***concrete*** and ***abstract*** meant very different things, despite being indiscriminately used, since, in his opinion, nothing is more concrete than a line, a color, or a surface.

Another art movement, the "Circle and Square", appeared in Paris in 1929 lead by the Belgian critic and artist Michel Seuphor (1901-1999) and by the Uruguayan painter Joaquim Torres-Garcia (1875-1949). In a periodical called *Circle and Square*, these artists published a manifesto proposing the regrouping of constructive artists, which included Wassily Kandinsky, Fernand Leger, Jean Arp, Le Corbusier, and others.

In Brazil, modern art started its penetration in the art world at the end of the 1940 decade, essentially with the creation of the São Paulo Museum of Modern Art in 1948 by Francisco Matarazzo Sobrinho. Its first director was Léon Degand, a Frenchman art critic who was a strong advocate of abstract art. The museum was officially inaugurated in 1949 with the exposition named "Do figurativismo ao Abstracionismo" ("From Figurativism to Abstractionism"). After several disturbing years, when it was even disbanded, the São Paulo Museum of Modern Art was reopened in 1969 at the Ibirapuera Park.

In Rio de Janeiro, a Museum of Modern Art was also established in 1948 at the then "Ministério de Educação e Cultura" (Department of Education and Culture), and substituted, in 1952, by a new museum in a portion of the borough of Flamengo that was reclaimed from the sea. Sadly, a fire destroyed the building and its contents in 1978.

The year 1951 saw the first São Paulo Art Biennial with the participation of many Brazilian artists, including Waldemar Cordeiro, Antonio Maluf, Abraham Palatnik and Ivan Serpa. Several artists were awarded international prizes, such as:

-Max Bill (1908-1994) won the International Grand Prize for Sculpture.

-Abraham Palatnik, who was born in Natal, Brazil, surprised the jury with a kinechromatic work that did not fit in any of the categories of the exposition, namely, painting, sculpture, etching, or drawing.

-Ivan Serpa won a Young Painter award for his work "Forms".

Still in 1951, in São Paulo, Samson Flexor (1907-1971), born in Bessarabia, Imperial Russia, founded the Atelier "Abstração" (Abstraction). Him and his students participated in the second art biennial in São Paulo.

In 1952, the poet Décio Pignatari, with Haroldo de Campos, Augusto de Campos and several others, created the "Noigandres Group", while Waldemar Cordeiro, Geraldo de Barros, Luiz Sacilotto, Lothar Charoux, Anatol Wladyslaw, among others, created the "Ruptura Group", which had its first art exposition in December 1952 in the São Paulo Museum of Modern Art.

Another group, called "Grupo Frente", appeared in Rio de Janeiro in 1954 with the participation of Ferreira Gullar and Mario Pedrosa and other artists such as Ivan Serpa, Aluisio Carvão, Décio Vieira, Carlos Val, Lygia Clark, Lygia Pape, João José da Silva Costa. Other artists, such as Helio Oiticica, César Oiticica, Franz Weissmann, Abraham Palatnik, and Rubem Ludolf, to name a few, also joined that group.

Art was experiencing a period of evolution during the 1950 decade and, in 1959, during an exposition at the Rio de Janeiro Museum of Modern Art a group of artists wrote a Neo-Concrete art manifesto involving painting, sculpture, etching, poetry, and prose. Such a group had the participation of Ferreira Gullar, Amílcar de Castro, Franz Weissmann, Lygia Clark, Lygia Pape, Reynaldo Jardim, and Theon Spanúdis.

Although the themes Concrete and Neo-concrete art are only briefly mentioned in this book, they should be studied in detail by those who are art aficionados since such themes are of fundamental importance to the art world and, in particular, to the evolution of the arts in Brazil. We call the reader attention to this because images of a number of paintings are presented in this book without any artistic analysis of them, and some readers might think that the geometric forms in those works were used without a specific purpose for them. It is necessary to grasp the real meaning and the message embedded in the work of the artists.

We pay tribute to some of the artists who participated in the initial phase of the concrete art in Brazil by showing images of their work in Figures 1.4 to 1.13.

Figure 1.4 “Composição” - Luiz Sacilotto - Oil on asbestos cement - 1948

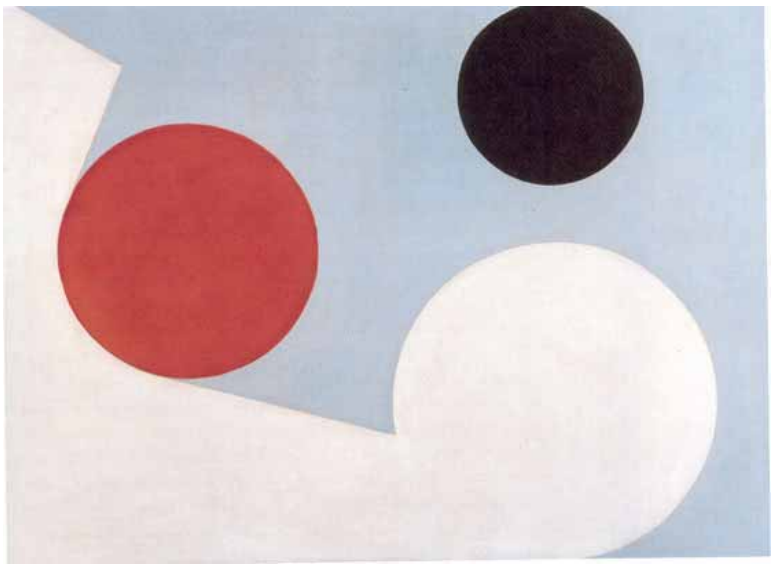

Figure 1.5 “Formas” - Ivan Serpa - Oil on canvas - Young Painter Acquisition Prize, First São Paulo Art Biennial - 1951

Figure 1.6 “Composição em Vermelho e Preto” - Aluisio Carvão - Oil on canvas - 1950 decade

Figure 1.7 “Equação dos desenvolvimentos com círculos” - Antonio Maluf - 1951

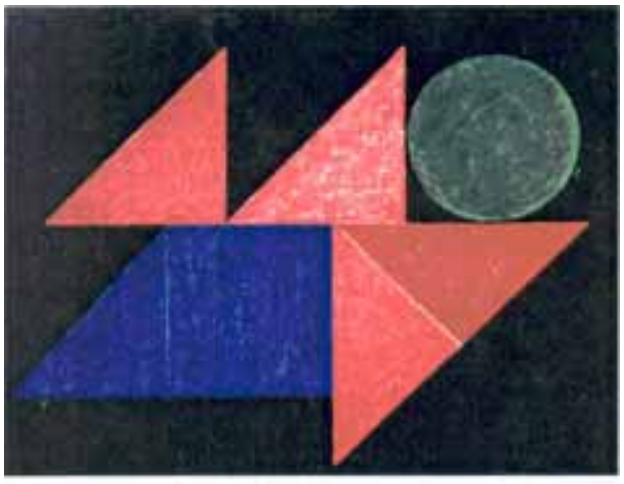

Figure 1.8 Untitled - Decio Vieira - 1980 decade

Figure 1.9 “Movimento Contra Movimento” - Geraldo de Barros - Enamel on Kelmite - 1948

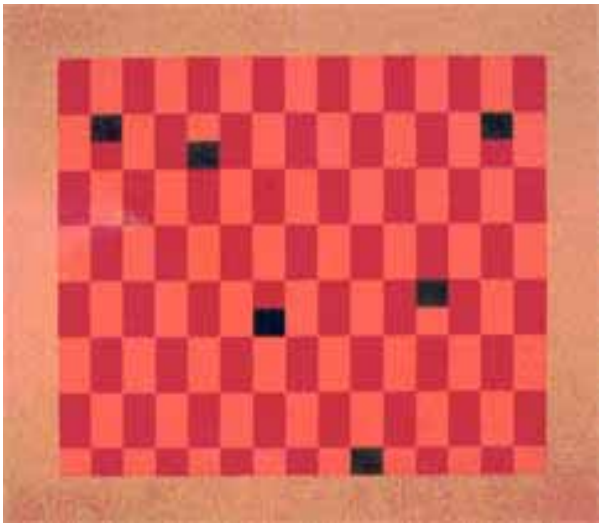

Figure 1.10 “Grupo Frente” - Helio Oiticica - Gouache on cardboard - 1955

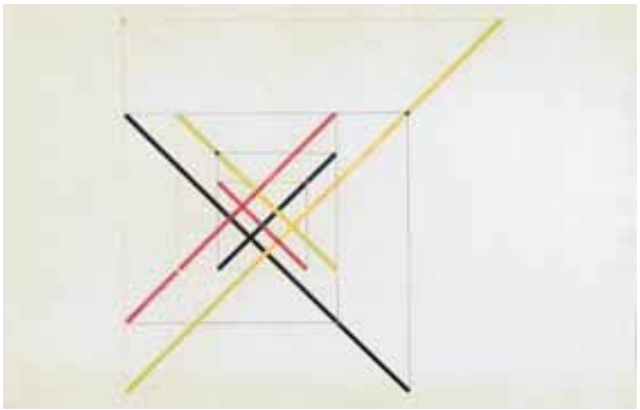

Figure 1.11 “Composição” - Lothar Charoux - Chinese ink on paper - 1958

Figure 1.12 “Assimetria Resultante de Deslocamentos Simétricos” - Rubem Ludolf - gouache on paper - 1955

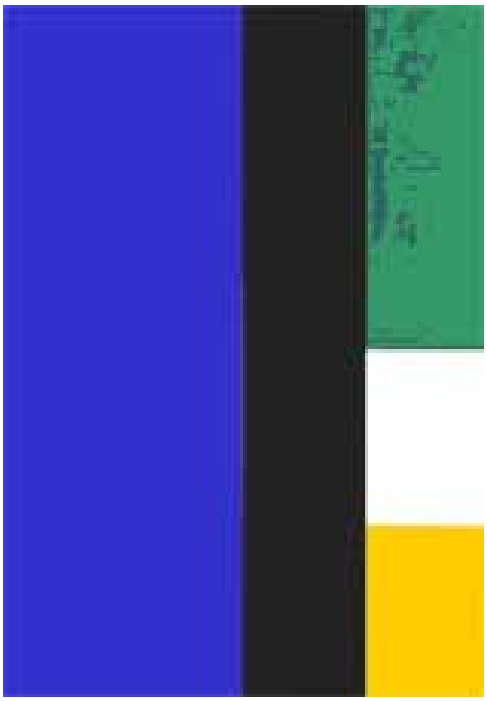

Figure 1.13 “Divisão pelas metades” - Almir Mavignier - Serigraph - 2005

The images presented above are only a few examples of concrete art by some Brazilian artists. Many other artists, including Alfredo Volpi, Dionísio Del Santo, Franz Weismann, Hércules Bersotti, Lygia Clark, Lygia Pape, Samson Flexor, Waldemar Cordeiro, Willys de Castro, contributed to that phase of the art in Brazil.

# 2

## Planar geometric forms

**Figure 2.1 Euclid**

## 2.1 Euclid of Alexandria

Euclid, one of the most prominent mathematicians from antiquity, was born in Greece, although the details of his birth and death are not known with certainty. It is believed that he went to Alexandria, in Egypt, around 300 BC, where he started a school of mathematics and created his most important work. Actually, almost all that is known about Euclid was compiled by Proclus (412-485) hundreds of years later, which accounts for the lack of accurate information about him.

A famous work by Euclid is "**The Elements**", which is a 13-volume treatise covering several areas of mathematics. Although it is not known which parts of that work were his own, all of its format and structure is attributed to him.

"The Elements" was translated to Latin and to Arabic and, with its first printing in 1482, became the most important source for mathematical studies at that time, and the most publicized work in the Western world, second only to the Bible.

## 2.2 The axiomatic method of Aristotle

Euclid used the so-called axiomatic method created by Aristotle, one of Plato's students, to develop the material in "The Elements", which contains all the fundamentals of what is known as ***Euclidean Geometry***. The axiomatic method consists in a systematic way for developing a theory starting from fundamental, or primitive, concepts that are observed and intuitively accepted. Next, facts accepted as true without the need of any demonstration are listed. These are called axioms (or propositions), and must be obeyed for developing a theory. Based on primitive concepts and on the axioms, one then demonstrates the theorems that are the base for the theory.

Although not adequate for all fields of science, the axiomatic method is widely accepted and used in mathematics. The method is illustrated in Figure 2.2.

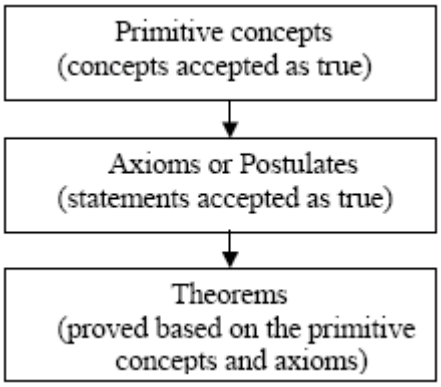


Figure 2.2 Axiomatic method

Note that the phrase ***Euclidean Geometry*** was emphasized at the beginning of this section. Without going into specific details, we point out that Euclid used only five postulates in his work and, based on them, proved over 400 theorems that were included in his book. Euclid, and others who succeeded him, tried unsuccessfully to prove the same theorems using only four postulates.

It was only in the 16$^{th}$ century that it became clear, due to the work of the German mathematician Carl Friedrich Gauss (1777-1855) and others such as Wolfgang Bolyai (1775-1856) and Nikolai Ivanovich Lobachevsky (1792-1856), that an alternative geometry could be consistently defined without the fifth postulate. This led to what is known as "**Non-Euclidean Geometry**". This may simply sound as a mathematical curiosity, but that is exactly what happened with the so-called "**Hyperbolic Geometry**", which was used by Einstein when developing his general relativity theory.

We are not going to study Euclid's work in depth, but only present its basic elements in the next section so that the most important geometric forms used in the arts can be better understood.

**2.3 Primitive concepts of Euclidean Geometry**

The primitive concepts of Euclidian geometry are the point, the straight line, and the plane. The **point** is a dimensionless element, meaning that it has no length, depth, or height. Imagine we touch the white surface of a sheet of paper with the tip of a pencil, and then keep sharpening the pencil until we have what we would call a point on the surface. If we were able to carry such an experiment to its limit, the point would be invisible! Clearly, no matter how much we sharpen the pencil we would never be able to have a point since the marking on the paper will always have a finite dimension regardless of how small it may be. We then realize that a point is, in reality, an abstraction that exists only in our imagination, although in everyday language the point is always thought of as something real. For example, looking at a very small star in the sky, or at a satellite in its orbit, one normally associates them with a point in space because what is seen from such a large distance is extremely small. Under such circumstances, it is important to keep in mind that this is merely a simplified way of describing some objects.

The next primitive concept of Euclidean Geometry is the **straight line**, which is characterized by having only one dimension. When we look at a straightened string, at the edge of a ruler, or at a straight railroad track, for example, we have the notion of what is called a straight line. Like the point, the straight line is also an abstract entity.

The third and last primitive concept is the **plane**, which is characterized by having two dimensions, namely length and width. One would think of a plane when looking at the surface of a table or the surface of a calm lake. In painting, for example, the canvas is our working plane.

It is important to stress again that the point, the straight line, and the plane are idealized entities, since in nature we only find shapes that may closely resemble them. Even in engineering and architecture, when drawings are made using a ruler, a square, and a compass, or using computer graphics in these more modern times, we can never really draw a perfect segment of a straight line or a perfect circle since imperfections would be apparent if the drawing is seen either through a powerful lens or through a microscope. However we unconsciously perceive the drawings made with a ruler or with a square as a straight-line segment, and our working table as a plane.

Figure 2.3 show points and straight lines drawn on the plane of the page of the book.

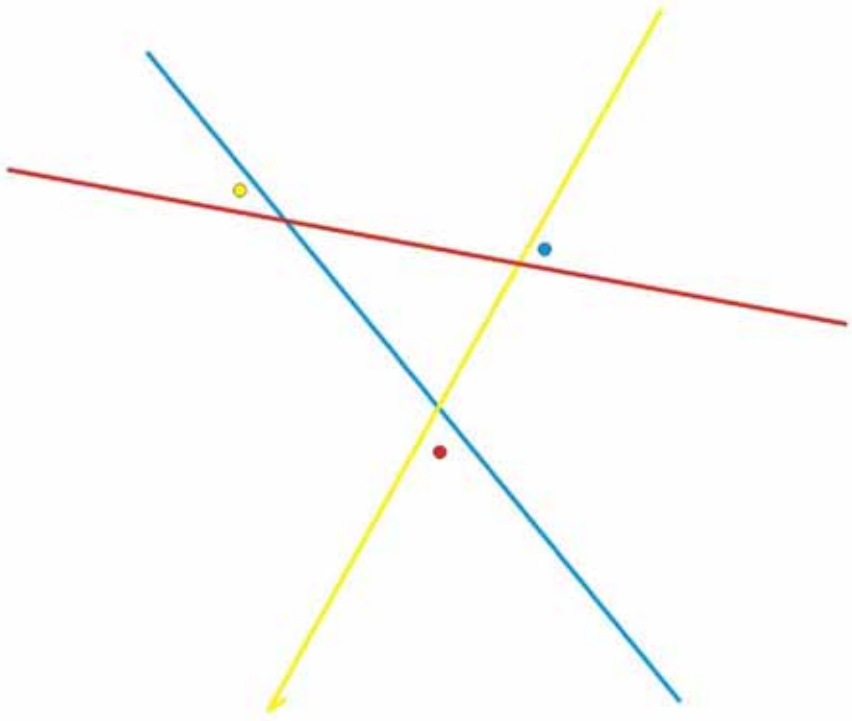

Figure 2.3 Points, straight lines, and plane - Hamello

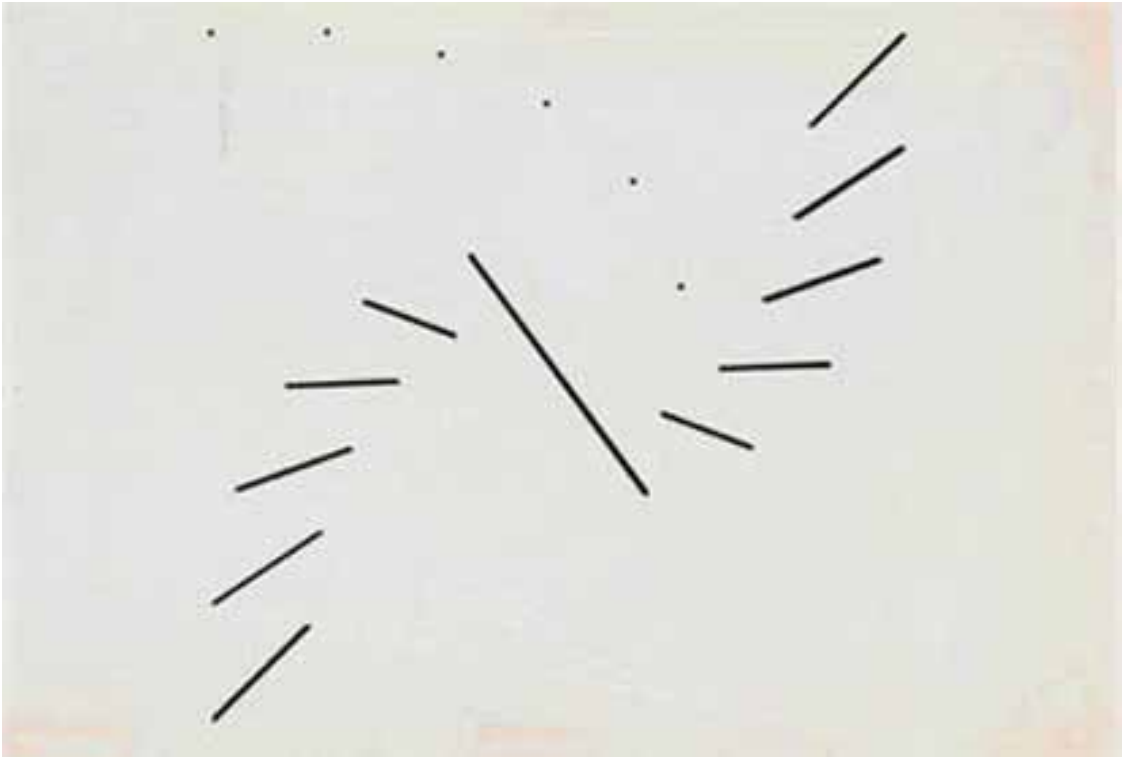

Figure 2.4 “Composição” - Lothar Charoux - 1958 - Chinese ink on paper

A close observation of any geometric form can disclose that all of them involve the primitive elements presented above. For example, angles are entities generated by two intersecting straight lines. They are classified as right, acute, or obtuse angles when they are equal, less than, or greater than 90 degrees, respectively. This is illustrated in Figure 2.5. The figure also shows the particular names that pairs of straight lines are universally referred to. Those that intersect are 90 degrees are called perpendicular lines, those that intersect at either acute or obtuse angles are called secants, and those that do not intersect are called parallel lines.

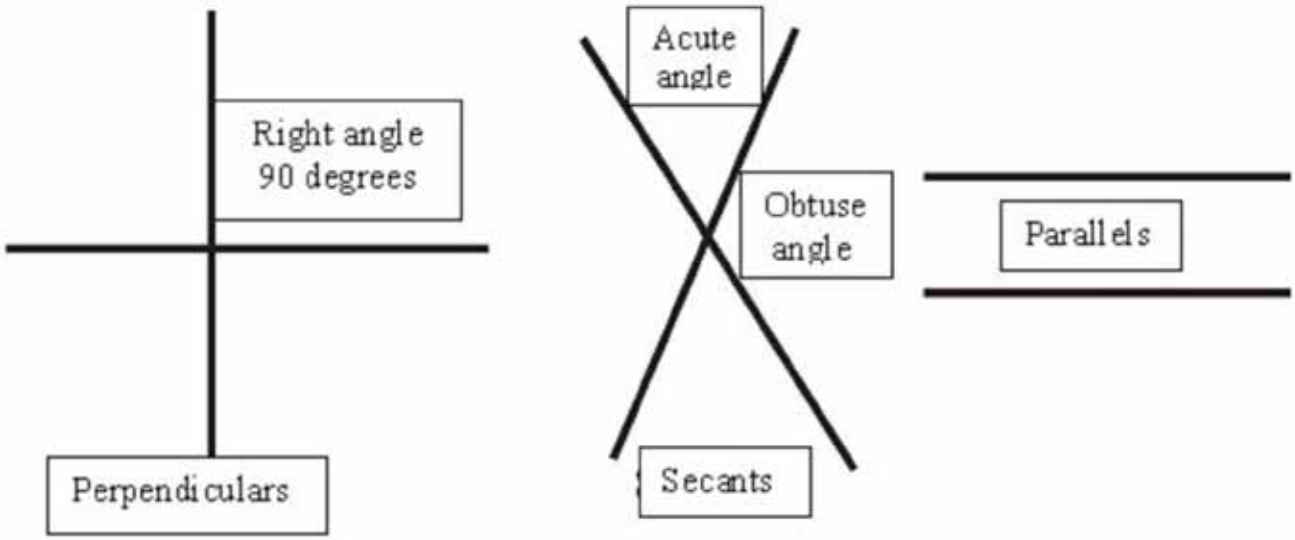


Figure 2.5 Relative positions of straight lines

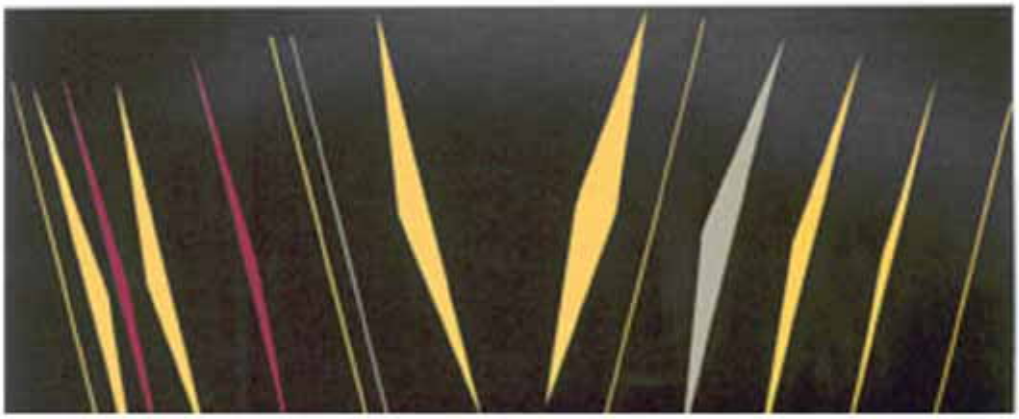

Figure 2.6 “Faixas Ritmadas” - Ivan Serpa - Oil on plate - 1953

## 2.4 Polygons

### 2.4.1 Definition

Polygons are closed planar figures bounded by three or more line segments. The line segments are called the polygon sides, and the points where they intersect are called the vertices. We have a broken line when the line segments do not form a closed figure. Several polygons and a broken line are shown in Figure 2.7.

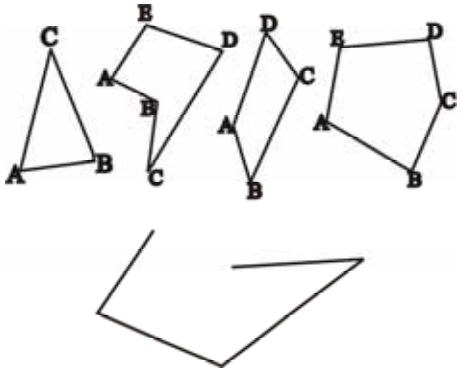


Figure 2.7 Example of polygons and of a broken line

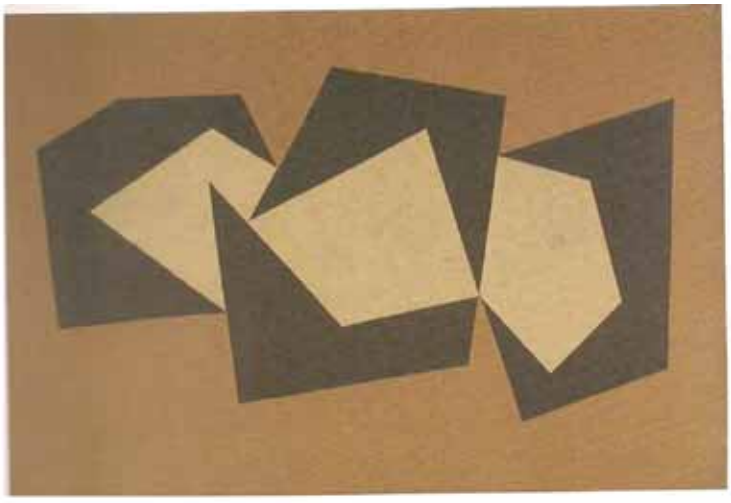

Figure 2.8 “Mineral” - Paiva Brasil - Collage on paper - 1957

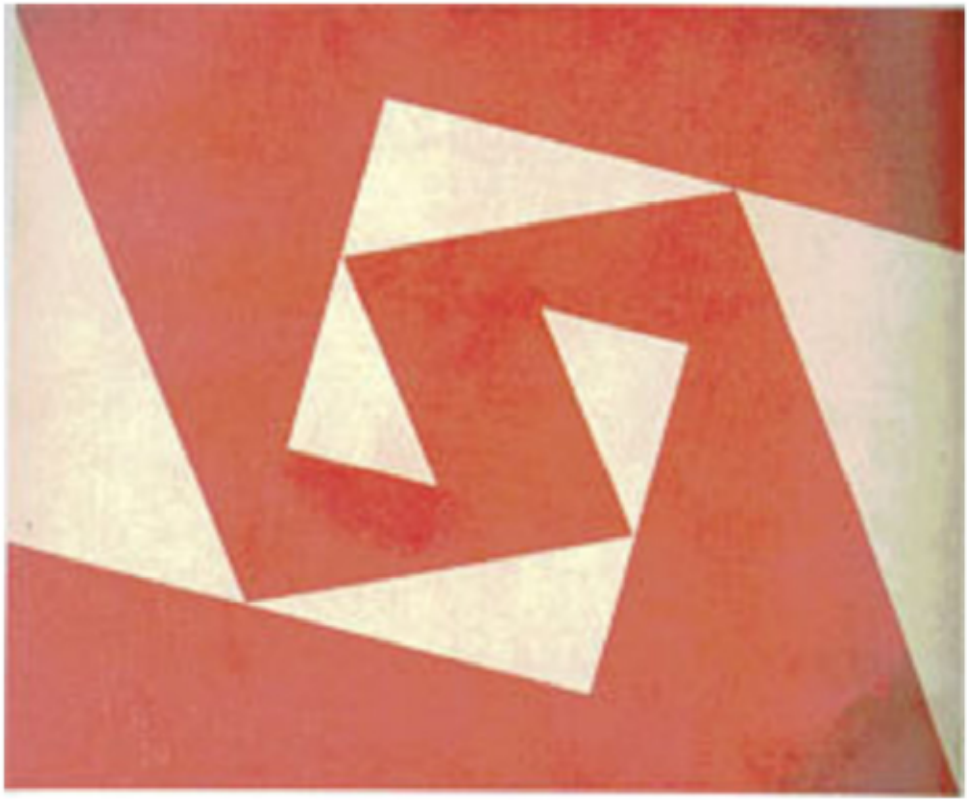

Figure 2.9 “Clarovermelho” - Aluisio Carvão - Oil on canvas - 1959

### 2.4.2 Polygon types

Polygons may be convex or concave. A polygon is said to be convex when the extension of any of its sides does not intersect any other side of the polygon. Otherwise it is said to be concave. That is, a concave polygon has dimples, as shown in Figure 2.10.

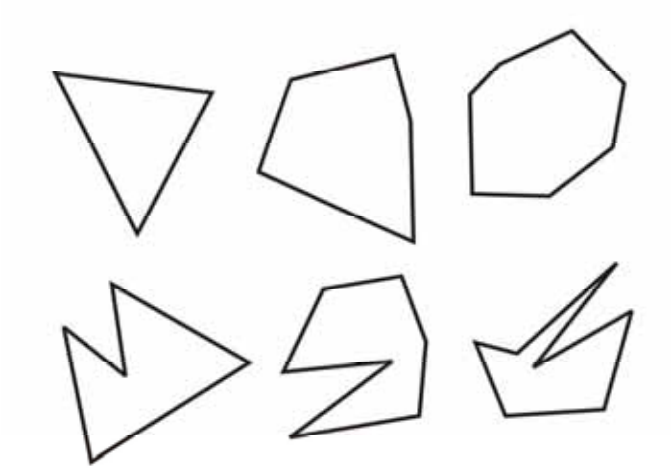

Figure 2.10 Convex (upper three) and concave (lower three) polygons

Polygons may also be classified as regulars and irregulars. A polygon is said to be regular when all its sides have the same length and all internal angles are the same, and irregular otherwise. Figure 2.11 shows examples of regular polygons.

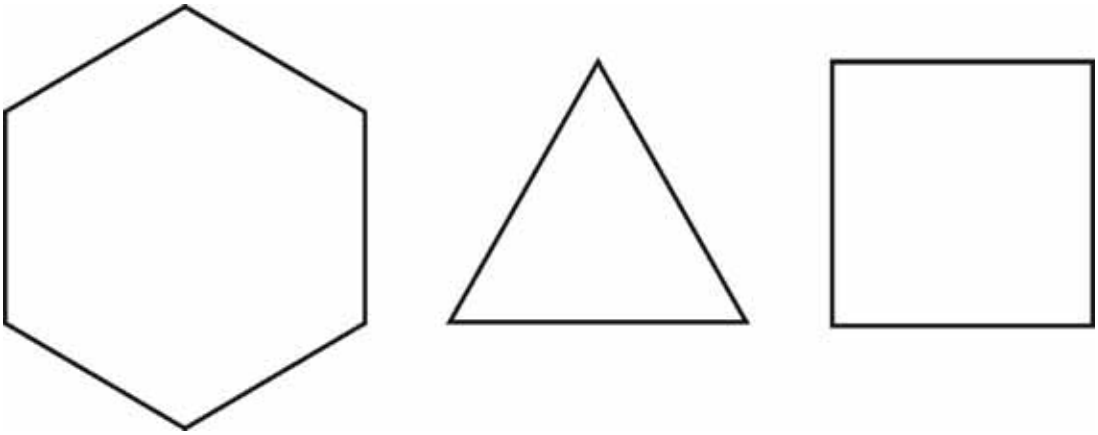

Figure 2.11 Regular polygons

Figure 2.12 shows a painting with polygons.

Figure 2.12 Untitled - Ivan Serpa - Oil on Canvas - 1952

### 2.4.3 Polygon names

Polygons are named according to its number of sides, such as Triangle (3 sides), Quadrilateral (4 sides), Pentagon (5 sides), Hexagon (6 sides), Heptagon (7 sides), Octagon (8 sides), Nonagon (9 sides), Decagon (ten sides), etc.

### 2.4.4 Triangles

Triangles are 3-sided polygons, which have special names according to the length of its sides: scalene when the three sides are unequal, isosceles when two sides are equal, and equilateral when the three sides are equal. If the internal angles of the triangle are considered, instead of the length of its sides, they are classified as acute, when all angles are less than 90 degrees (called acute angles), obtuse, when one angle is greater than 90 degrees (called an obtuse angle), and rectangular, when it has a 90 degrees angle (called a right angle). Figure 2.13 shows examples of these triangles.

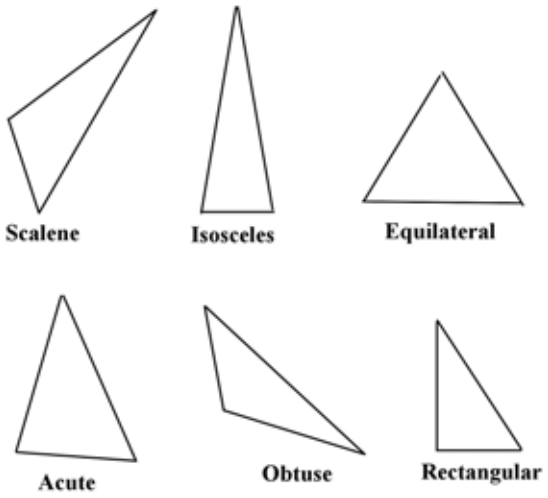


Figure 2.13 Triangle types

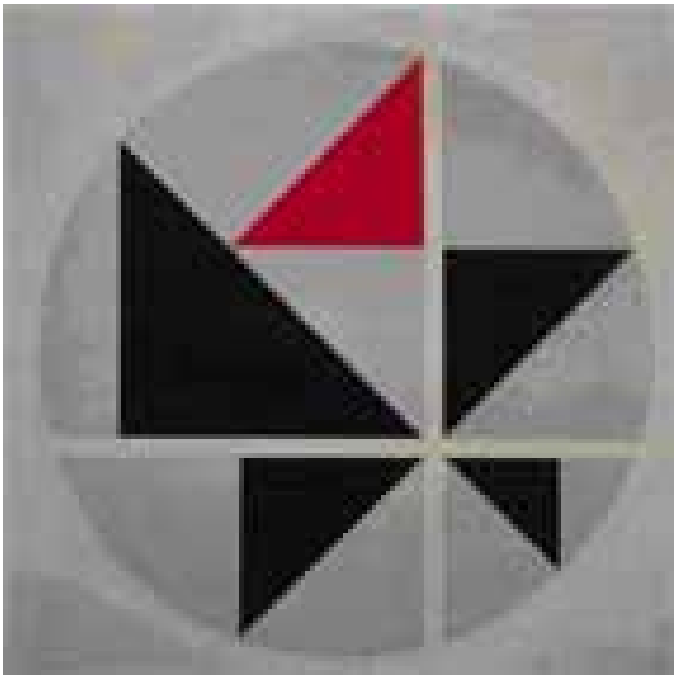

Figure2.14 “Tema Triangular” - Aluisio Carvão - Oil on Canvas - 1957

Figure 2.15 “Composição” - Lothar Charoux - 1958 - Chinese ink on paper

### 2.4.5 Quadrilaterals

As mentioned earlier, a quadrilateral is a polygon with four sides. Some quadrilaterals have special names and deserve special attention due to their importance.

#### 2.4.5.1 Parallelograms

A parallelogram is a quadrilateral whose opposite sides are parallel to each other. This is shown in Figure 2.16, where sides AB and CD are parallel to each other, the same occurring with sides AD and BC.

Figure 2.16 Example of a parallelogram

Parallelograms are known by special names depending on the length of its sides and of its internal angles. Rhombus, when all sides are equal; rectangle when the angles are equal to 90 degrees; and square when its sides are equal and the angles are 90 degrees. This is illustrated in Figure 2.17.

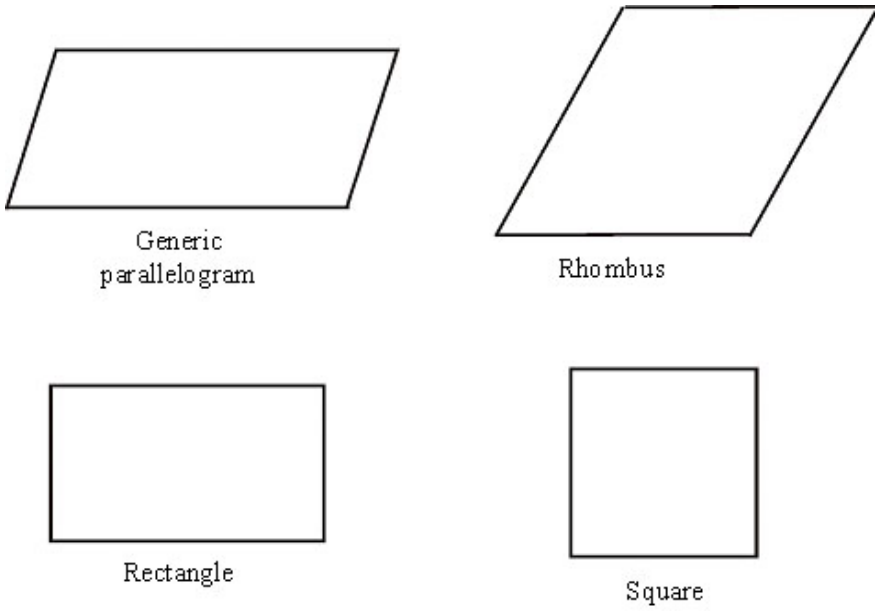


Figure 2.17 Types of parallelograms

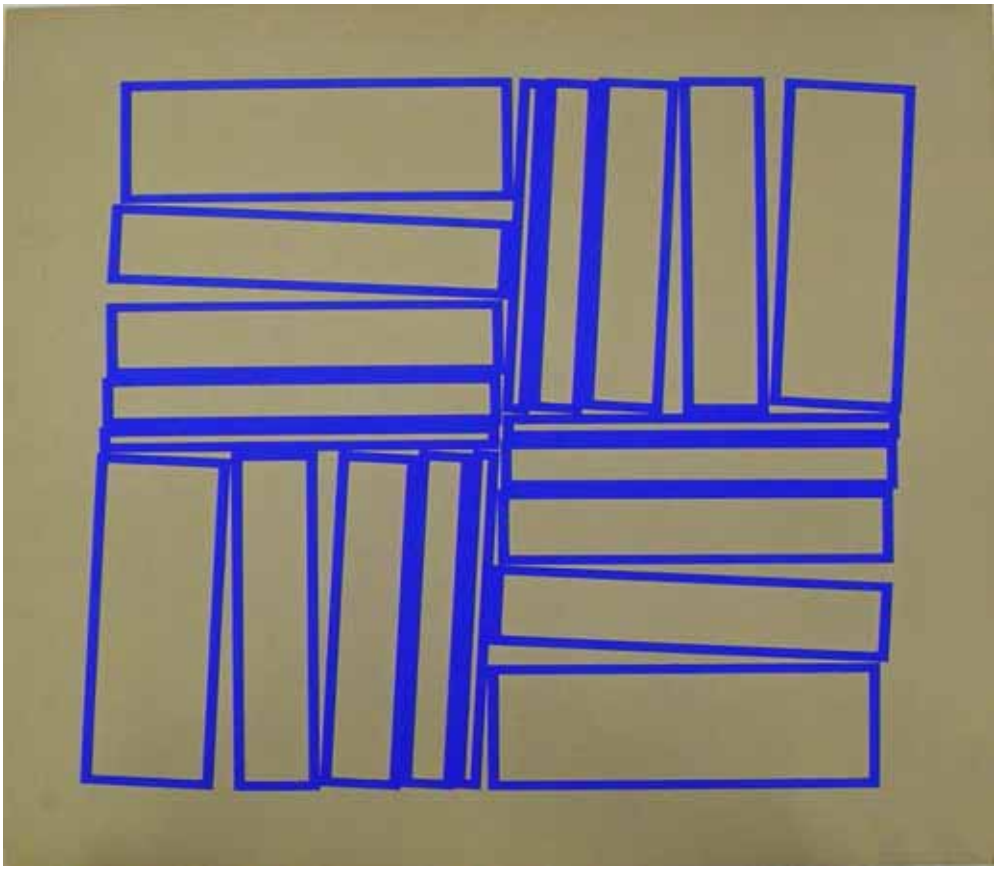

Figure 2.18 “Metaesquema” - Helio Oiticica - Goauche on pape - 1958

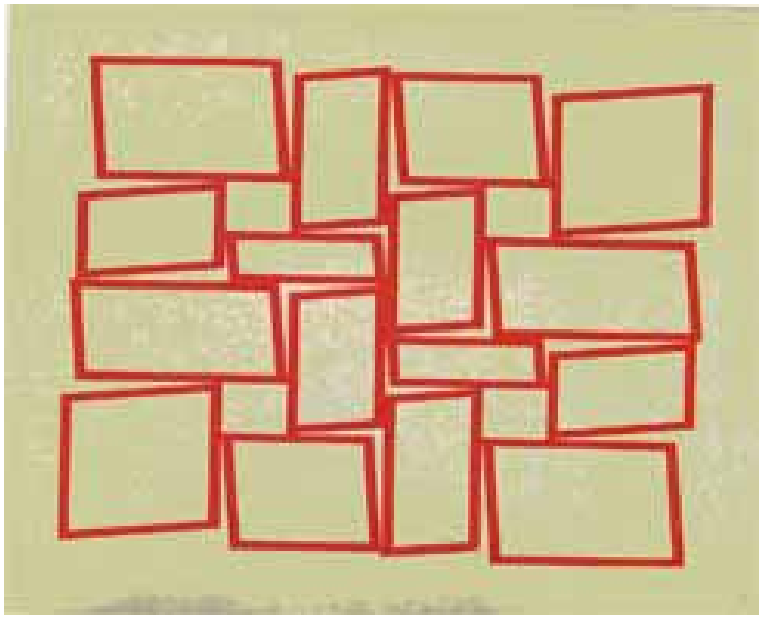

Figure 2.19 “Metaesquema” - Helio Oiticica - Gouache on paper - 1958

Figure 2.20 Untitled - Geraldo de Barros - Assembly on laminated plastic - 1983

Figure 2.21 shows an ancient type of Chinese puzzle called the TANGRAM. It consists of seven polygons forming a perfect square, made of cardboard, wood, or any other material. The objective of the puzzle is to construct a figure using all seven pieces without overlaping them. The image shown in the lower part of Figure 2.21 is one example out of the many that are possible.

Figure 2.21 The Tangram

### 2.4.5.2 Trapezoids

A trapezoid is a quadrilateral with only two parallel sides. The parallel sides are called the bases of the trapezoid (referred to as larger and smaller bases), and the perpendicular distance between them is the height of the trapezoid. Specific names are given to some trapezoids, depending on their sides and internal angles. A trapezoid is said to be isosceles when the length of its non-parallel sides are equal, and rectangular when two of its sides are perpendicular to each other. Figure 2.22 shows the different types of trapezoids.

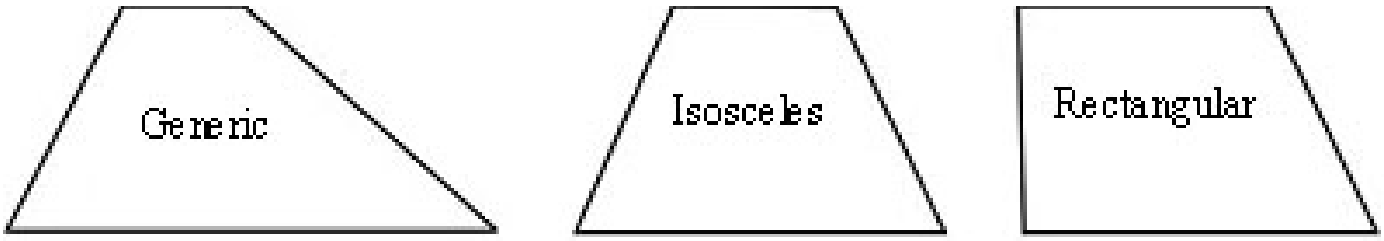


Figure 2.22 Trapezoid types

## 2.5 Circumference and circle

A circumference is a set of points that are equidistant from a single point O, called the center, as shown in Figure 2.23. The distance between O and any point of the circumference is called the radius (which is R in Figure 2.23).

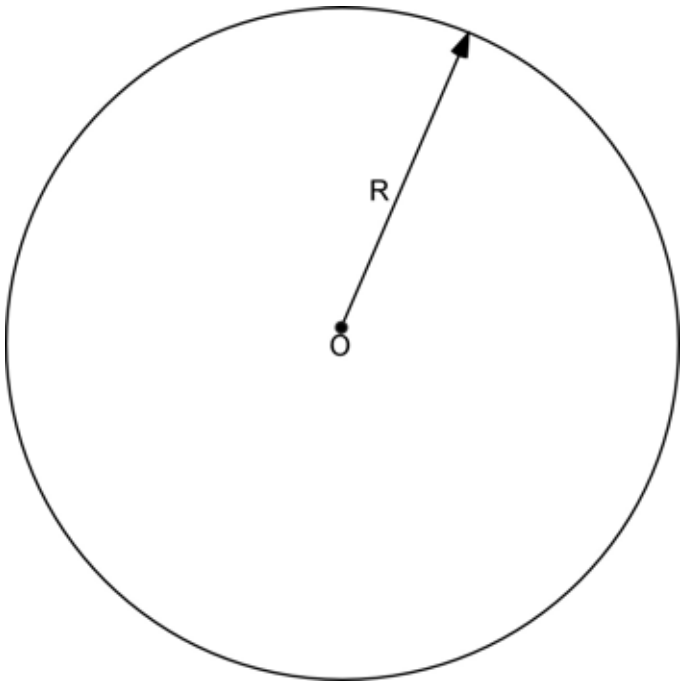


Figure 2.23 Circumference

The circle is the internal part of the plane bounded by the circumference. In other words, the circumference is a line, which is measured, of course, in units of distance such as meters. The circle is a surface, and its area is measured in are units, such as square meters ($m^2$).

Figure 2.24 Untitled - Ivan Serpa - Oil on canvas - 1965

Figure 2.25 “Equação dos desenvolvimentos com círculos” - Antonio Maluf - 1951

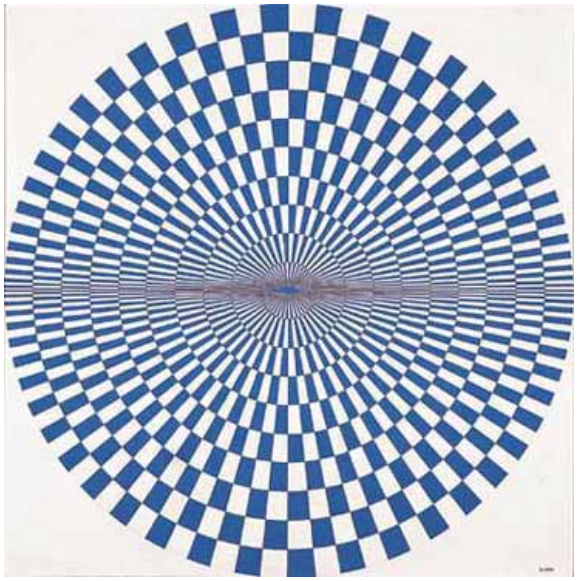

Figure 2.26 “Concreção 7961” - Luiz Sacilotto - Tempera on canvas/wood - 1979

**2.6 Inscribed and circumscribed polygons**

A convex polygon is inscribed on a circumference when all its vertices lie on a circumference, and circumscribed when all its sides are tangential to a circumference. This is illustrated in Figure 2.27 with a pentagon; the figure on the left is an inscribed pentagon, and the one on the right is a circumscribed pentagon.

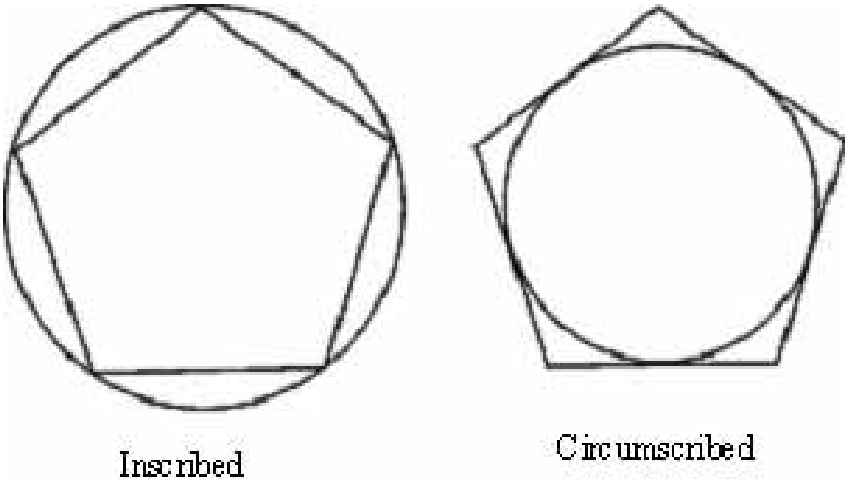


Figure 2.27 Inscribed and circumscribed polygons

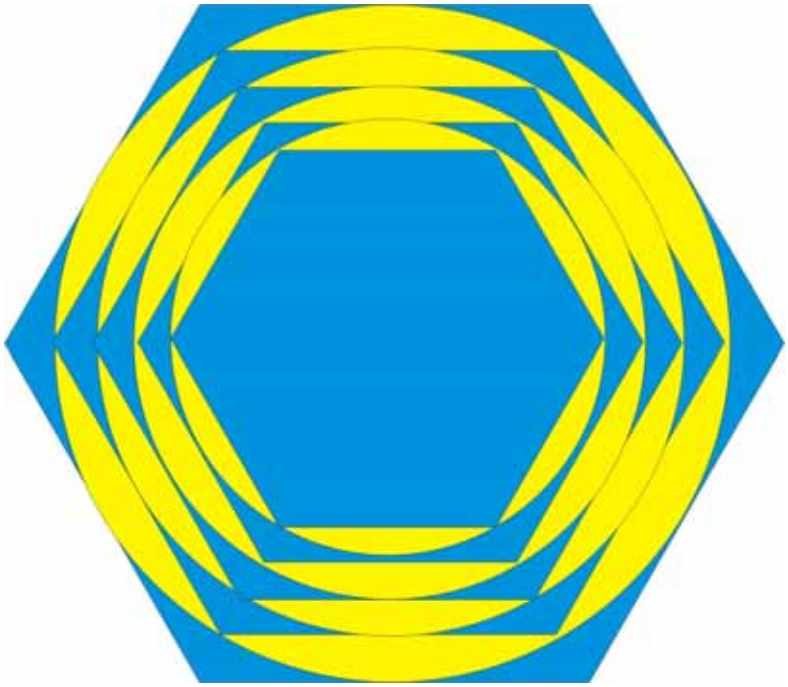

Figure 2.28 Inscribed and circumscribed - Hamello

Circles have been and still are extensively used in the plastic arts. We point out to the reader the works of a number of artists including Paul Klee (1879-1940), Robert Delaunay (1885-1941), M. C. Escher (1898-1972), Victor Vasarely (1906-1997), Wassily Kandinsky (1866-1944), Paul Nash (1889-1946), to name a few.

**2.7 Stars**

A number of interesting compositions can be created with figures known as geometric stars and, for that reason, a brief study of them is presented in the sequel. There are in nature forms that are called stars, such as the starfish, in the animal kingdom, and the leaves and flowers of several plants, in the vegetal kingdom. However, only geometrically perfect stars, as exemplified in Figure 2.29, are considered in the study presented in the sequel.

Figure 2.29 18-point star - Hamello

A practical method to draw a star involves starting with a polygon with the same number of vertices as the number of points we want the star to have. For example, we can draw a 5-point star by starting with a pentagon; the five vertices of the pentagon will become the five points of the star. Now, choose one of the vertices and connect it to another vertex of the pentagon, but always skipping one in between, and continue this process until you return to the vertex you started with. The result is a 5-point star, as shown in Figure 2.30. The resulting star was colored to stress the fact that we started with a pentagon and obtained another pentagon near the center of the figure, in an inverted position relative to the original one. We could now use the vertices of that internal pentagon to draw another star, and continue such a process to obtain any number of stars, one inside the other.

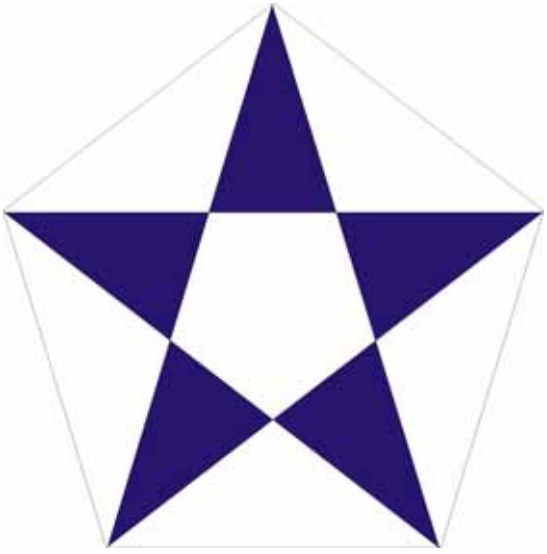

Figure 2.30 5-point star

A 6-point star can be drawn starting with a hexagon, as illustrated in Figure 2.31.

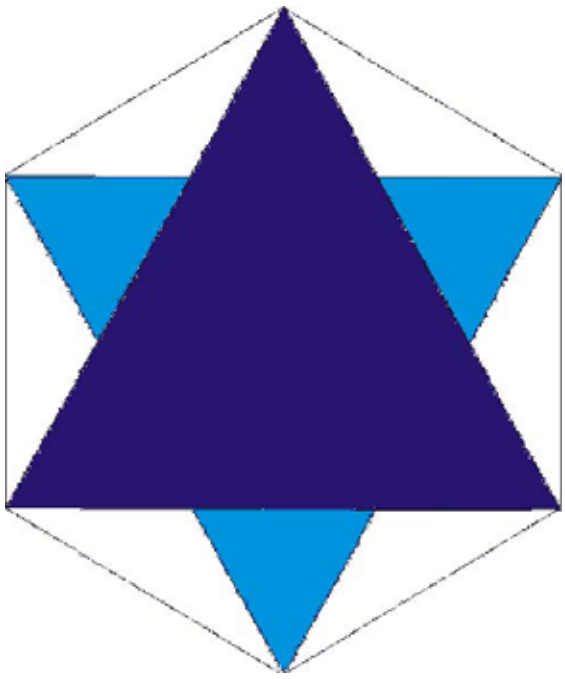

Figure 2.31 6-point star

Actually, what we obtained in Figure 2.31 were two inversely positioned triangles, instead of a continuous star. Such a star is called the Star of David**,** and it is the Jewish symbol studied in Chapter 9.

A 7-point star can be drawn starting with a heptagon (i.e., seven vertices), with two possibilities since one or two vertices may be skipped. This yields the two stars shown in Figure 2.32, which are colored for artistic reasons. Notice that another heptagon is obtained in such a process, allowing for the drawing of other stars indefinitely.

Figure 2.33 shows 8, 9, 10, 11, and 12-point stars that can be drawn using a similar process. We leave it to the reader to analyze these drawings. A composition based on stars is shown in Figure 2.34.

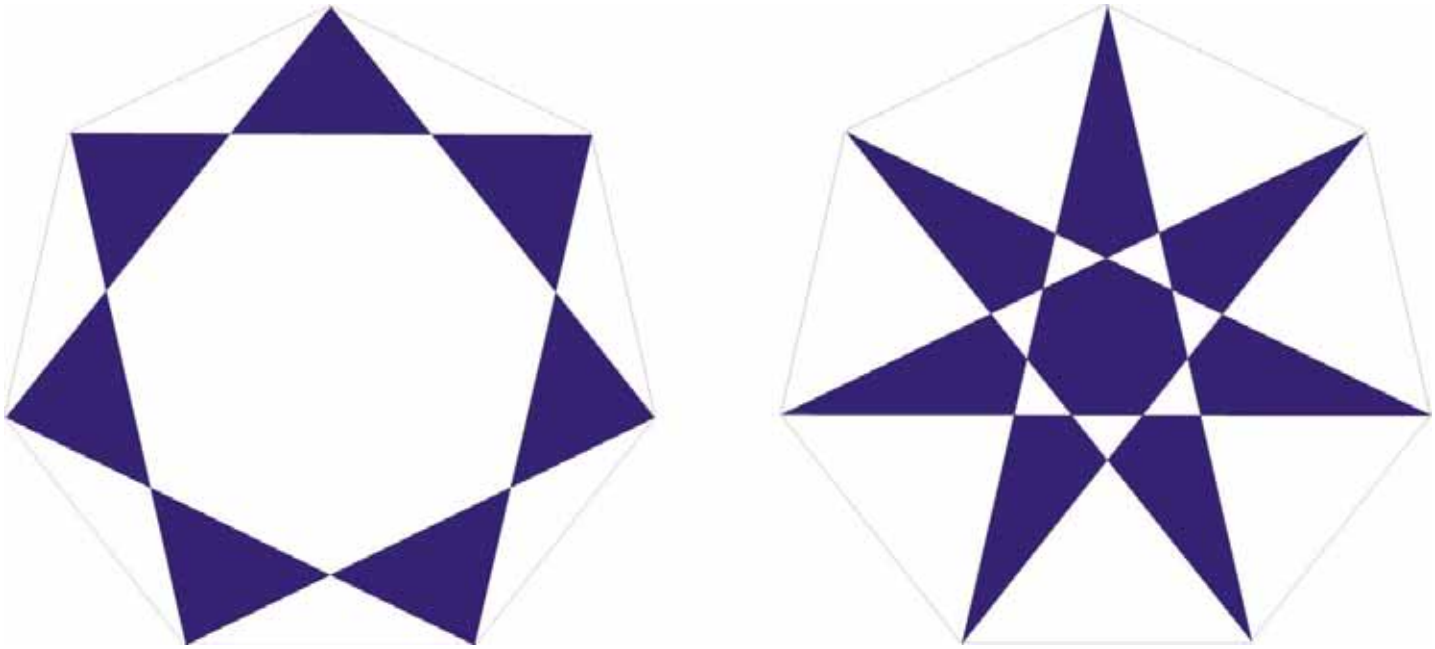

Figure 2.32 7-point stars

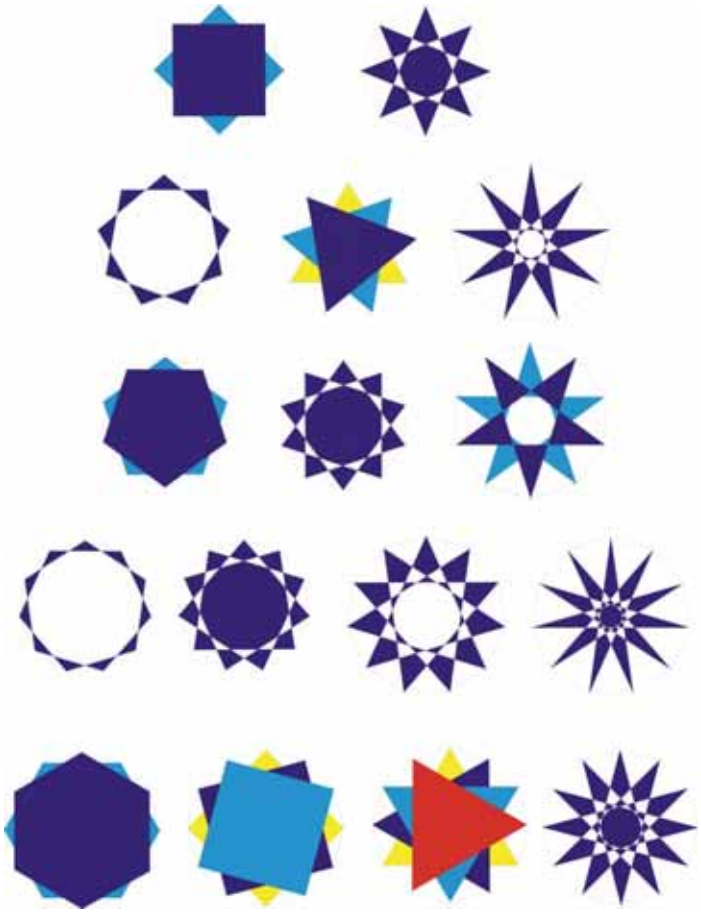

Figure 2.33 Stars with 8, 9, 10, 11, and 12 points

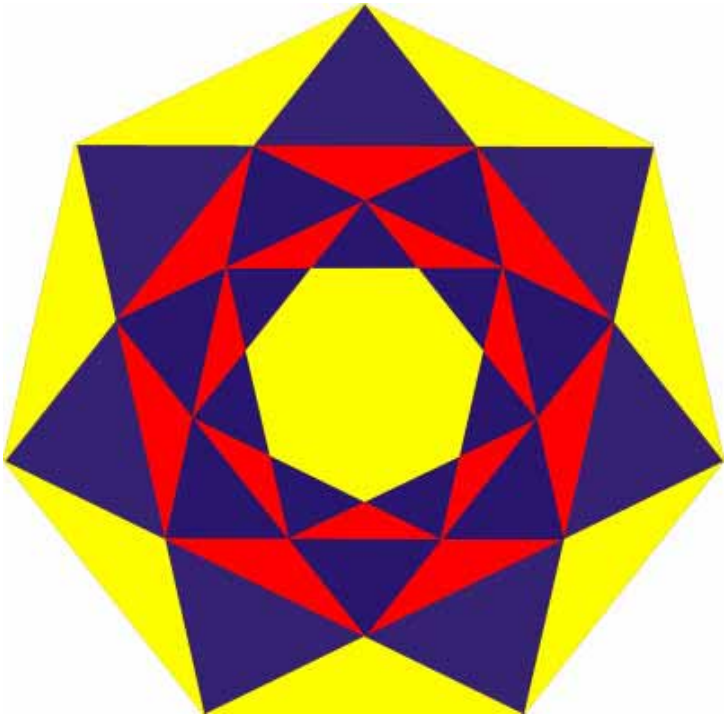

Figure 2.34 Composition with 7-point stars - Hamello

## 2.8 Rosaceae

Geometric rosaceae (plural of rosacea) are figures that remind us of roses, but have a symmetric structure as illustrated in Figure 2.35.

Figure 2.35 Examples of rosaceae

Any rosacea may be drawn using classical instruments such as the square and the compass. However, by observing the rosacea on the upper right corner of Figure 2.35, it can be seen that it can be drawn by turning the gray circle shown in Figure 2.36 about a fixed point (which is the center of the rosacea) to positions 1, 2, 3, etc, up to position 10. Such a practical procedure is especially useful for those who are not very familiar with geometric constructions.

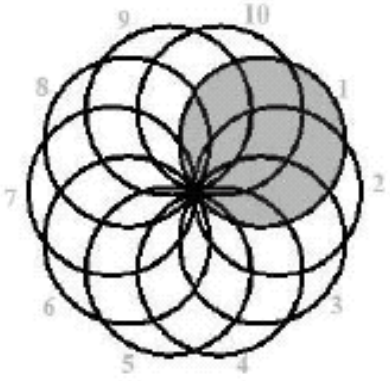


Figure 2.36 A practical method for drawing a rosacea

Simple rosaceae may be combined to form more complex structures. Such is the case of the so-called "Flower of Life", for example, found in the Temple of Osiris at Abydos in Egypt, and shown in Figure 2.37.

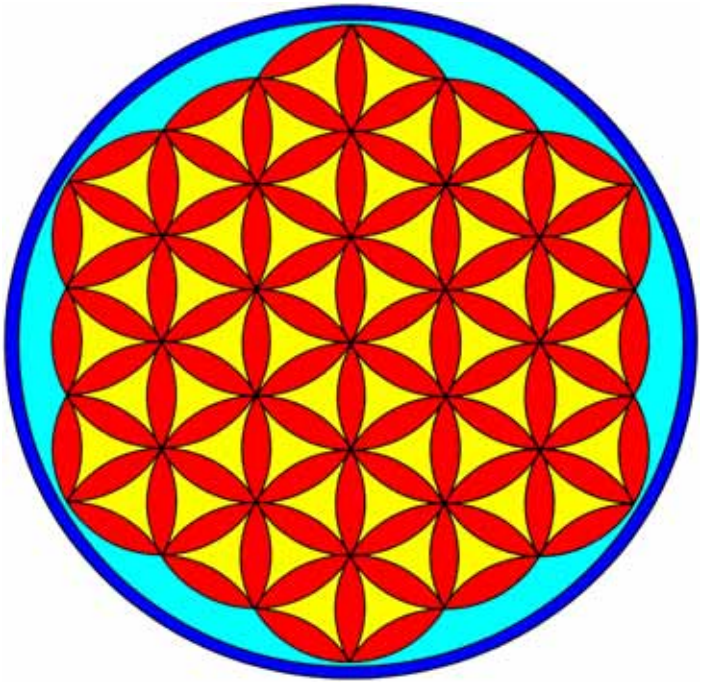

Figure 2.37 Flower of Life - Temple of Osíris

Figures 2.38 and 2.39 show two compositions by Steve Frisby.

Figure 2.38 “Star Burst” - Steve Frisby - Acrylic on canvas

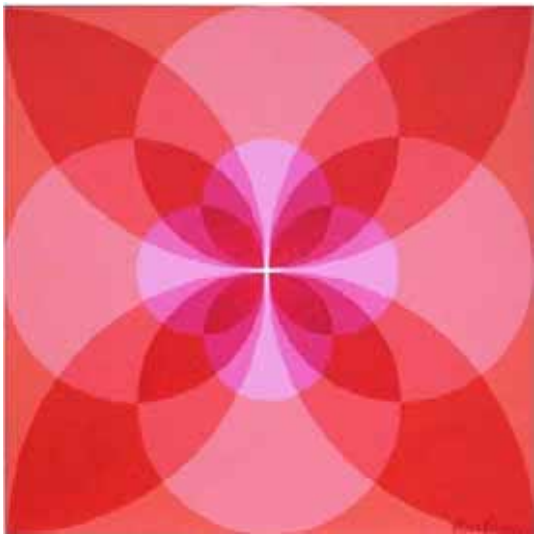

Figure 2.39 “Circles Meet” - Steve Frisby - Acrylic on canvas

Rosacea were used in the construction of stained glass of several Cathedrals around the world, and a beautiful example is in the Notre Dame Cathedral, in Paris, at the banks of the river Seine (see Figure 2.40). Construction of the Notre Dame Cathedral started in 1163, during the reign of Louis VII, and Pope Alexander III laid its cornerstone. Its stained glass windows were built with colored glass, with the color of the glass introduced during the manufacturing process. The manufacturing of the glass, and the manner it was cut for assembling the rosacea, are examples of the refined techniques of that time. It is interesting to note that the image shown in Figure 2.40 is seen from inside the Cathedral, which explains why only the window is illuminated by the outside light.

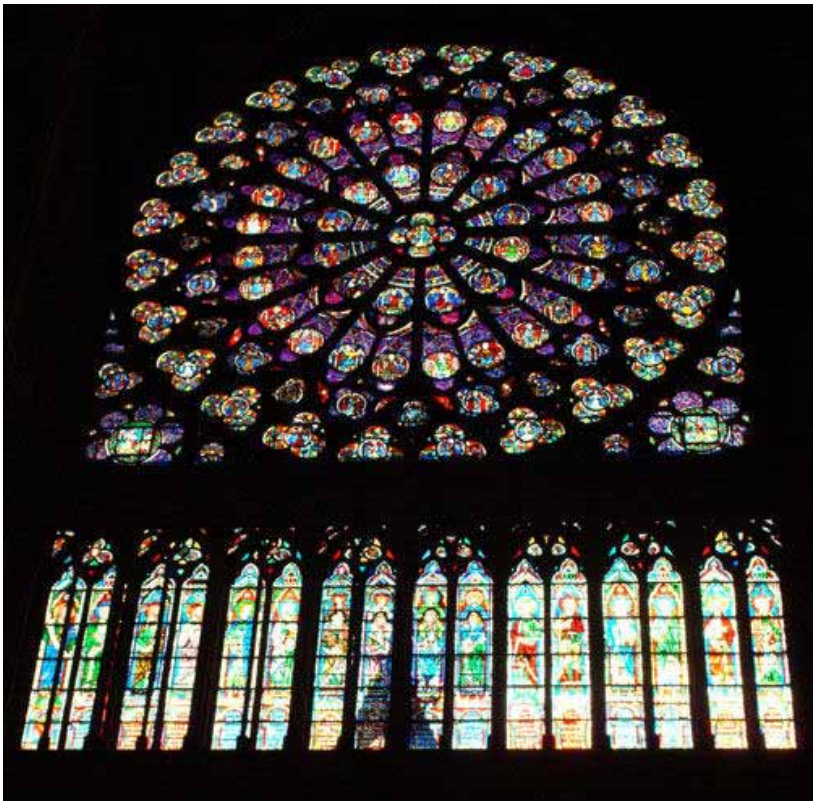

Figure 2.40 Rosacea at the Notre Dame Cathedral in Paris

## 2.9 Spirals

The spiral is a plane curve that winds around a fixed center, with the distance from the center to points in the spiral continuously increasing or decreasing. There are several types of spirals and many mathematicians and scholars studied them. Some spirals are named after those scholars, such as Archimedes spiral, Bernoulli spiral, Fermat spiral, and many others. Only the first two mentioned above are shown here, with an illustration on how they appear in our lives.

### 2.9.1 Archimedes spiral

Such a spiral was investigated by Archimedes (287 BC-212 BC), one of the most important mathematicians from antiquity. Archimedes was born in what is now the City of Syracuse, in the island of Sicily. He is well known for the famous phrase: “Give me a lever and a fulcrum and I will move the world”. Archimedes’s spiral, probably the best known of the spirals, is characterized by the fact that the distance between each adjacent loop of the spiral is constant. It is shown in Figure 2.41.

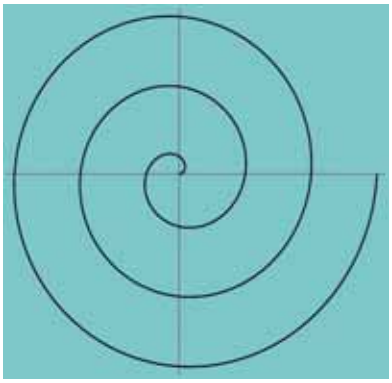

Figure 2.41 Archimedes spiral

### 2.9.2 Bernoulli’s spiral

Bernoulli’s spiral, also known as logarithm or equiangular spiral, was investigated by the Swiss mathematician Jacob Bernoulli (1654-1705). He was fascinated by that spiral and requested that it be engraved on his grave tombstone with the Latin inscription ***Eadem mutata resurgo***, literally meaning "Although changed I shall rise again the same". This was an allusion to the fact that the angle bounded by the tangent at any point on the spiral, and the line from the origin of the spiral to that point (angle $\alpha$ in Figure 2.42), is a constant. This is an interesting property of that spiral. By observing Figure 2.42, it is seen that, unlike Archimedes spiral, the distance between any two adjacent loops of Bernoulli’s spiral is always increasing.

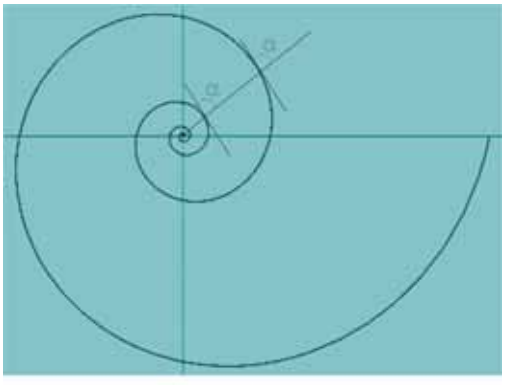


Figure 2.42 Bernoulli spiral

### 2.9.3 Spirals in our lives

Spirals are very common figures that are present in nature and in much of the work created by humans. They are an archetype that has been in existence since ancient civilizations, and a highly symbolic figure in many cultures.

Let us first observe an example from nature. Figure 2.43 shows a sea snail, known as nautilus. The upper portion of the figure shows the entire snail while the lower portion shows a transversal cut of the same, disclosing its internal structure in the form of a spiral.

Figure 2.43 The Nautilus shell

There are several examples in the vegetal world showing how spirals in harmony with nature. An interesting example is the spiraling distribution of seeds in the sunflower, shown in Figure 2.44.

Figure 2.44 Spirals formed by the sunflower seeds

Spirals are also seen elsewhere, in addition to the vegetal and animal kingdoms. When humans were able to observe the Cosmos more closely with the help of telescopes, it was seen that several galaxies, such as the Milky Way, where our solar system is located, and the Andromeda Galaxy, had a spiral form. Spirals have also appeared in buildings since antiquity. For example, the top of the Ionic Order Columns, built in ancient Greece, has two spirals connected to each other.

In the field of plastic arts, many artists used spirals in their work. Some of them are mentioned here, but it should not surprise anyone if any of the names mentioned below do not seem to be associated with their known work, as artists pass through distinct phases in their artistic life. Spirals were used in the work entitled ***Painter's Spiral Dance***, by the Croatian painter Boris Demur, presented in 1966 at the XXIII São Paulo International Biennial and included in the catalog for that Exposition. The painting called ***Swirlfish,*** by the Dutch artist Maurits C. Escher (1898-1972), shows fish moving along a double spiral. Another interesting work is the one called ***Sphere Spirals***, also by Escher, involving spirals on a spherical surface. Many more artists used spirals in their work, including Alexander Calder (1898-1976) and Joan Miró (1893-1983).

We finish this chapter by presenting in Figure 2.45 a festival of planar geometric forms, which the reader may want to use for identifying each of the forms present in the figure.

Figure 2.45 Festival of planar geometric forms - Hamello

# 3

## Polyhedrons

**Figure 3.1 Whoville - George Hart - Aluminum**

## 3.1 Introduction

Polyhedrons are solid figures bounded by plane polygons. Two examples of polyhedrons are shown in Figure 3.2: a cube and a truncated icosahedron.

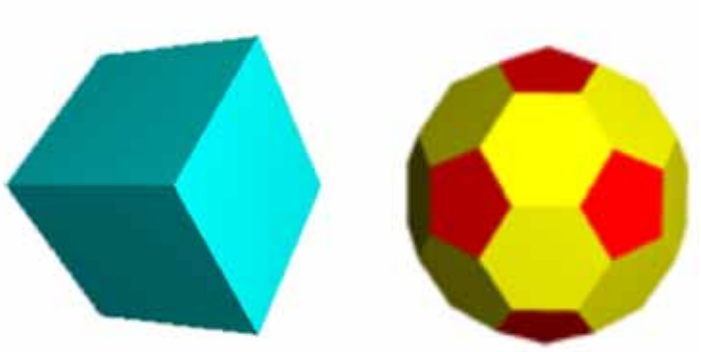

Figure 3.2 Cube and truncated icosahedron

The intersection of two faces of a polyhedron is called the **edge**, and the intersection of two edges is called a **vertex**. A cube has 6 faces, 12 edges, and 8 vertices, while a truncated icosahedron has 32 faces, 90 edges and 60 vertices.

As in polygons, polyhedrons also have distinct names depending on some of its characteristics. A polyhedron with 4 faces is called a tetrahedron; pentahedron, for 5 face polyhedrons; hexahedron, when it has 6 faces; heptahedron, for a 7 face polyhedron; octahedron, for an 8 face polyhedron, and so on. Some polyhedrons are best known by more common names such as the cube, which is a hexahedron with square faces.

A polyhedron is called **regular** when all of its faces are equal regular polygons, with the same number of faces intersecting at each of its vertices. The cube shown in Figure 3.2 is a regular polyhedron since all of its faces are equal regular squares, and three of its faces always intersect each other at each vertex. On the other hand, the truncated icosahedron, also shown in Figure 3.2, is not a regular polyhedron since it has two types of face, which are pentagons, shown in red in the figure, and hexagons, shown in yellow.

The piece shown in figure 3.3, made by Aluisio Carvão, is a nice example of how a simple geometric figure may be transformed into art.

Figure 3.3 “Cubocor” - Aluisio Carvão - Pigment and oil over cement - 1960

## 3.2 Convex and concave polyhedrons

The concepts of convex and concave polyhedrons are an extension of the same concepts associated with polygons discussed in Chapter 2. A polyhedron is said to be convex when the extension of the plane of any of its faces does not intersect any other face of the polyhedron. Otherwise it is said to be concave. Thus, a concave polyhedron exhibits "concavities". The polyhedrons shown in Figure 3.2 are convex, and the one shown in Figure 3.4 is concave.

Figure 3.4 Example of a concave polyhedron

## 3.3 Interesting families of polyhedrons

### 3.3.1 Platonic Solids

When studying polygons we saw that regular polygons may have any number of sides, 150 for example. However, polyhedrons are more restrictive. They may be constructed with any number of faces only if no restrictions are imposed on them. If restrictions are imposed, the number of possibilities is reduced. Such is the case, for example, when one requires that the polyhedron be regular and convex. It can be proved that it is possible to construct only five types of such polyhedrons, which is a fact known since Antiquity. Those five solids, known as **Platonic Solids**, are shown in Figure 3.5. Although they are named after Plato (428 BC-347 BC), who wrote about them in his philosophical work, they have been known well before his time.

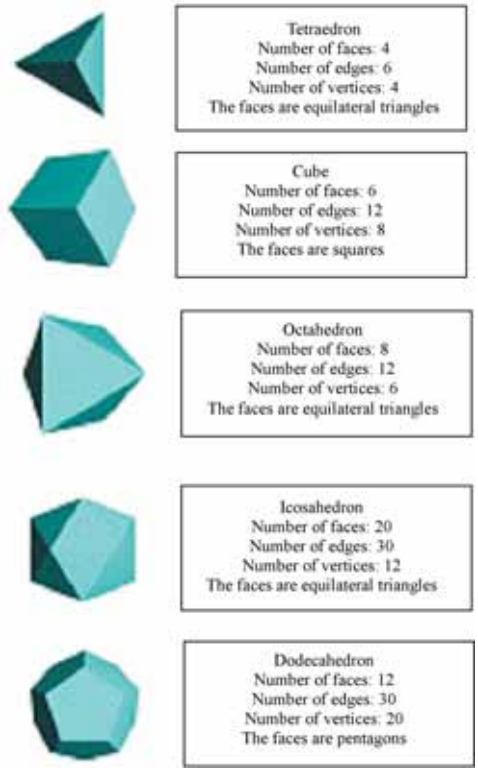


Figure 3.5 Platonic solids

### 3.3.2 Archimedean solids

Archimedean solids, named after Archimedes, are characterized by being convex polyhedrons whose faces are two or more types of regular polygons. There are thirteen of such solids, and they are shown in Figure 3.6.

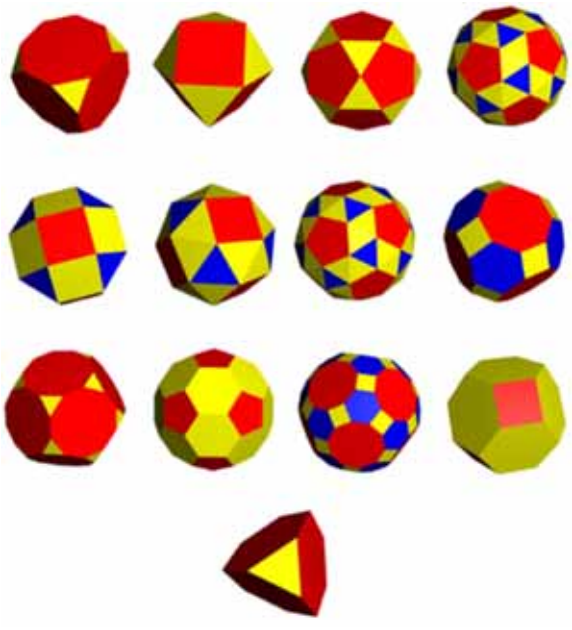

Figure 3.6 Archimedean solids

It is interesting to note that several of the Archimedean solids may be obtained by truncating (i.e., cutting off pieces) an appropriate Platonic solid at its vertices. For example, the solid shown in the top left part of Figure 3.6 may be obtained by truncating a cube at its vertices. For this reason, that particular solid is known as a truncated cube. Five of the Archimedean solids may be obtained by truncating the five appropriate Platonic solids.

Figure 3.7 shows a sculpture using polyhedrons.

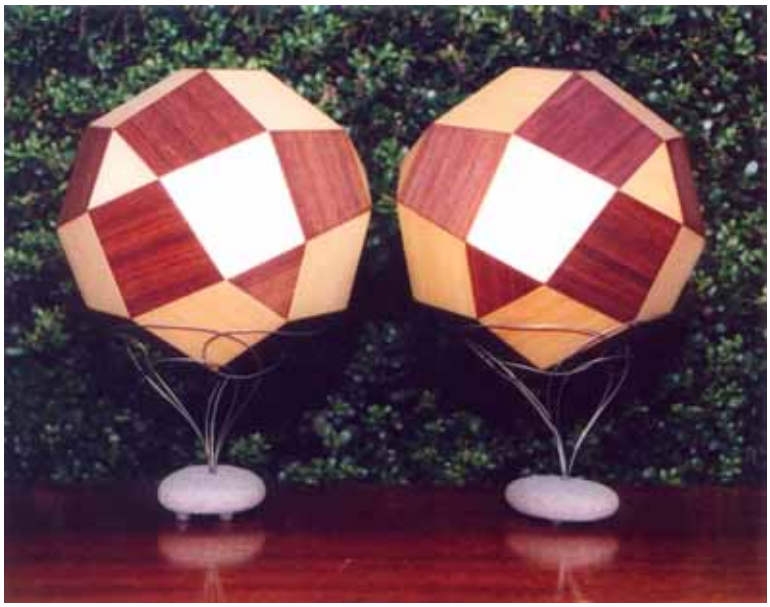

Figure 3.7 Yin and Yang - George Hart - Wood (Walnut and Basswood)

### 3.3.3 Star solids

Star solids are named as such because they resemble three-dimensional stars. One way to construct a star solid is to start with an appropriate polyhedron and extend the plane of its faces that do not have a common edge until they intersect to form a new polyhedron. This process is called stellation. Some polyhedrons may be transformed into star polyhedrons by using different planes in the process. To give the reader an idea of the importance of the stellation process we note that just the icosahedron can generate 58 different star solids. Figure 3.8 shows four possible stellations of the icosahedron, and Figures 3.9 to 3.11 show star solid sculptures.

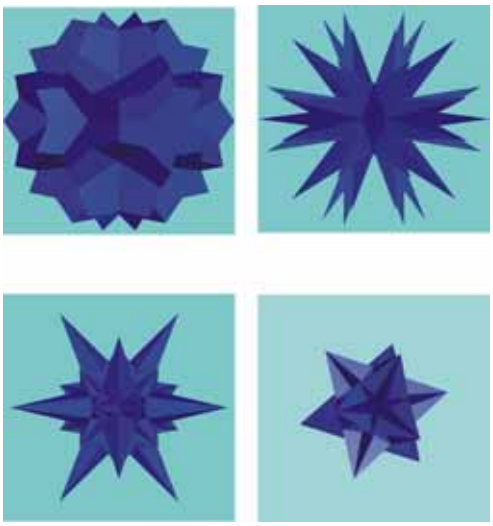

Figure 3.8 Examples of stellations with the icosahedron

Figure 3.9 “Compass Points” - George Hart - Wood (Cedar and Plywood)

Figure 3.10 ““Giri” - Tom Lechner – Wood (a variety)

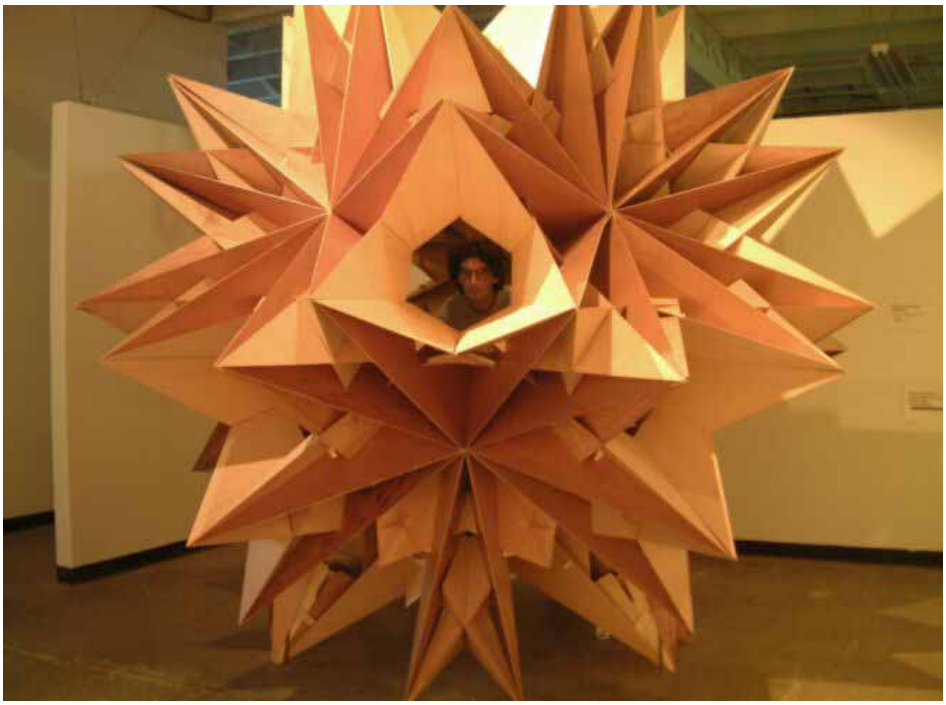

Figure 3.11 “Peekaboo” - Tom Lechner  - Wood (a variety)

### 3.4 Other polyhedrons

#### 3.4.1 Pyramid

A pyramid is a solid obtained by connecting a point, called the pyramid's vertex, to the vertices of a polygon. That polygon is called the base of the pyramid, and the pyramid is named after that polygon. Figure 3.12 shows three types of pyramids, namely, triangular, pentagonal, and square.

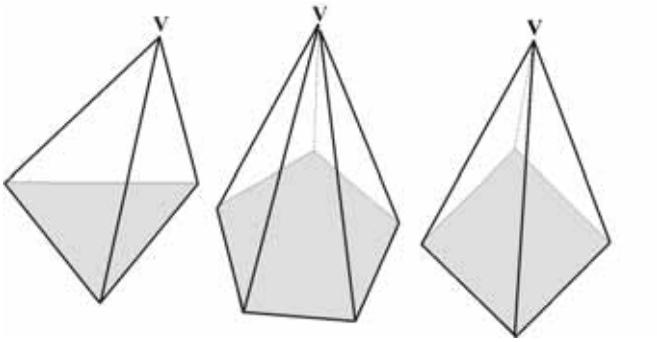


Figure 3.12 Pyramid types

The square pyramids of Cheops, Chephren and Micherinos, in Giza, Egypt, are examples of the best known of such solids. They are shown in Figure 3.13. Details of the construction of those pyramids, such as their dimensions, orientation, secret chambers, and other characteristics, can be found in a number of books and documents.

Figure 3.13 Pyramids of Cheops, Chephren, and Micherinos

The glass pyramid at the Louvre, designed by the architect I. M. Pei and shown in Fig. 3.14, is an example of a contemporary architecture using a pyramid. It was a source of considerable controversy at the time of its construction because, for many, it was an aberration to the Louvre architecture.

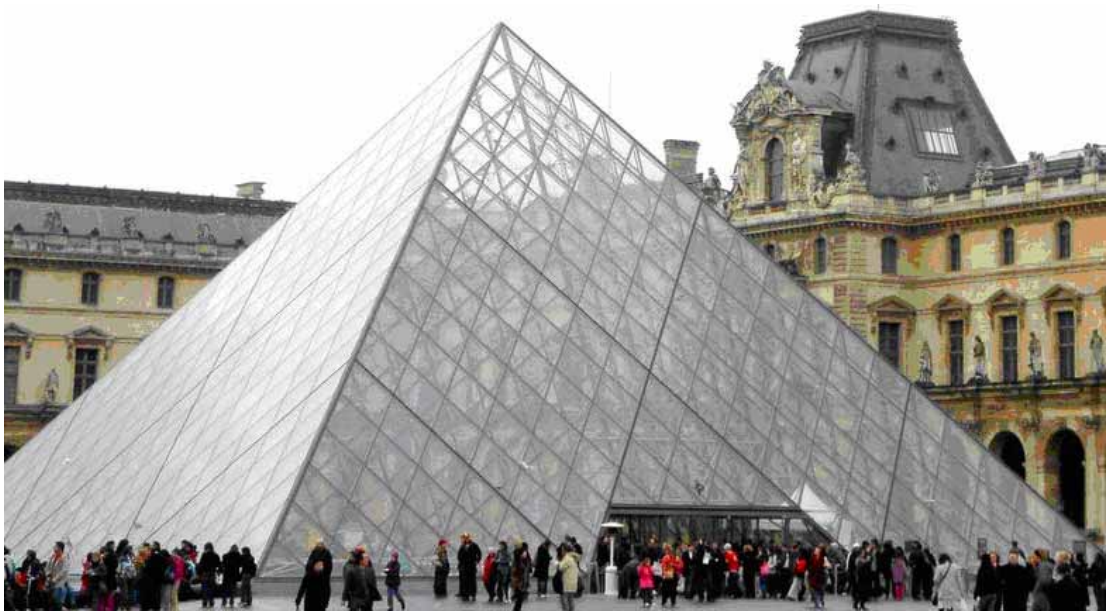

Figure 3.14 Pyramid at the Louvre

Figure 3.15 shows a contemporary architectural pyramid in Brazil, and Figure 3.16 is a painting by the author, based on an Egyptian theme.

Figure 3.15 The Peace Pyramid - “Legião da Boa Vontade”

Figure 3.16 Mysterious Egypt - Hamello

### 3.4.2 Truncated pyramid

A truncated pyramid, shown in Figure 3.17, is obtained by cutting a pyramid with a plane parallel to its base.

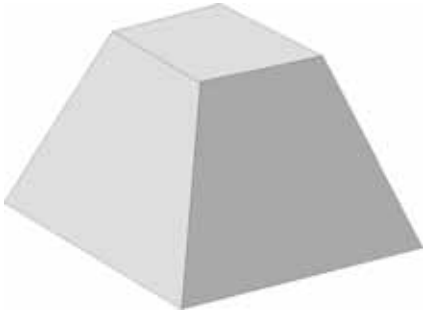

Figure 3.17 Truncated pyramid

### 3.4.3 Straight prism

A straight prism is a polyhedron with identical upper and lower faces, and whose lateral faces are rectangles An example is the hexagonal straight prism shown in Figure 3.18.

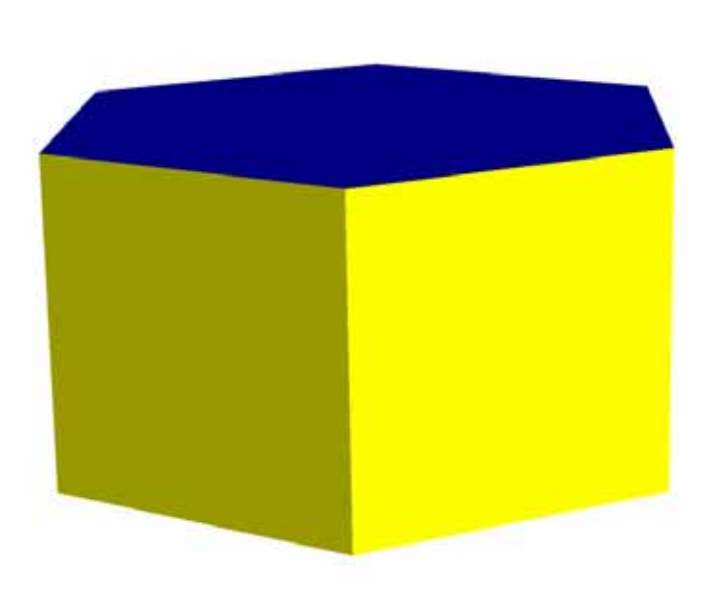

Figure 3.18 Hexagonal straight prism

Figure 3.19 shows an interesting sculpture made with prisms.

Figure 3.19 Untitled - João Galvão - Acrylic on wood - 1968/2003

**3.5 Polihedrons and the great masters**

Leonardo Da Vinci (1452-1519), one of the geniuses of the Renaissance, created many artistic works involving geometry, which was one of his passions. Luca Pacioli (1445-1514), who was a Franciscan Friar, used one of Da Vinci's works -- a series 60 figures with solids -- in his book "De Divina Proportione". Three of such figures are shown in Figure 3.20. It is interesting to note that in the upper two images shown in Figure 3.20 the bodies are represented by "solid edges", allowing a viewer to see through them and have a precise idea of what is in front and behind them. In the lower image in Figure 3.20 one cannot see either the interior of the body or what is behind it because it is represented as a massive solid. The general belief is that the idea of representing the body by solid edges is due to Da Vinci, although there is no proof of such.

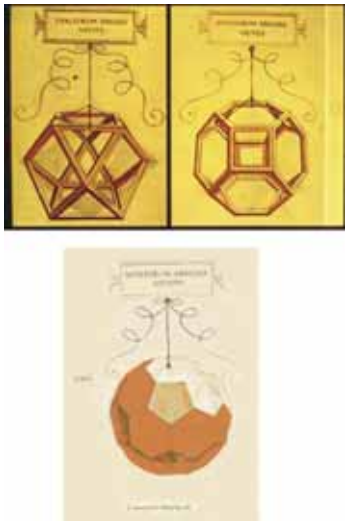

Figure 3.20 Da Vinci's works illustrating Pacioli's book

A painting of Luca Pacioli, with his geometrical instruments, is shown in Figure 3.21. The painting is attributed to Jocopo de Barbari (1440-1515) and illustrates the connection between the Renaissance and geometry. Two solids are evident in that painting: on the upper left corner one sees a "rhombic cube octahedron", made of transparent material and half full with a liquid; in the lower right corner there is a dodecahedron on top of what seems to be either a book or a box.

Figure 3.21 Luca Pacioli - Jacopo de Barbari - oil on canvas - National Gallery of Capodimonte - Naples, Italy - 1495

Figure 3.22 shows a work by the German artist Albrecht Durer (1471-1528), called Melancholia, studied by several authors in a number of art books and articles. It is an engraving where one can see a sphere and a polyhedron that may be a cube truncated at its upper vertex. As for the base of the polyhedron, it is not possible to know if the opposite vertex of the cube was cut off or if the cube penetrates its supporting surface. Such different possibilities allow for several interpretations of that work, justifying the great interest many have on it.

Figure 3.22 Melancholia - Albrecht Durer - Engraving - 1514

Other artists from the same period, such as Paollo Uccello (1397-1475), Piero della Francesca (1416/1420?), and Fra Giovanni (1387-1455), also used polyhedrons in their work.

**3.6 Polyhedrons and informatics**

The study of spatial geometric forms, such as polyhedrons, is a fascinating branch of mathematics. Computers made it possible to create many of such forms that are not only very difficult to draw by hand, but also require extensive and very involved mathematical calculations to generate them. There are special computer programs for generating and visualizing planar and spatial geometric forms, and some of them are listed in Chapter 10. To spark the reader curiosity, two computer-generated images are shown in Figures 3.23 and 3.24.

Figure 3.23 “Dodicosa” - Russel Towle

Figure 3.24 “Cetros” - Russel Towle

**3.7 Closing comments on polyhedrons**

A complete study of polyhedrons is a complex matter, and some of their forms are difficult to construct. In this book only some of the families of solids, such as Platonic, Archimedean, and a few star solids, were studied. It should be noted, though, that there are other types of solids that may be of interest to the plastic arts due to their beauty and exotic appearance.

# 4

## Other spatial figures

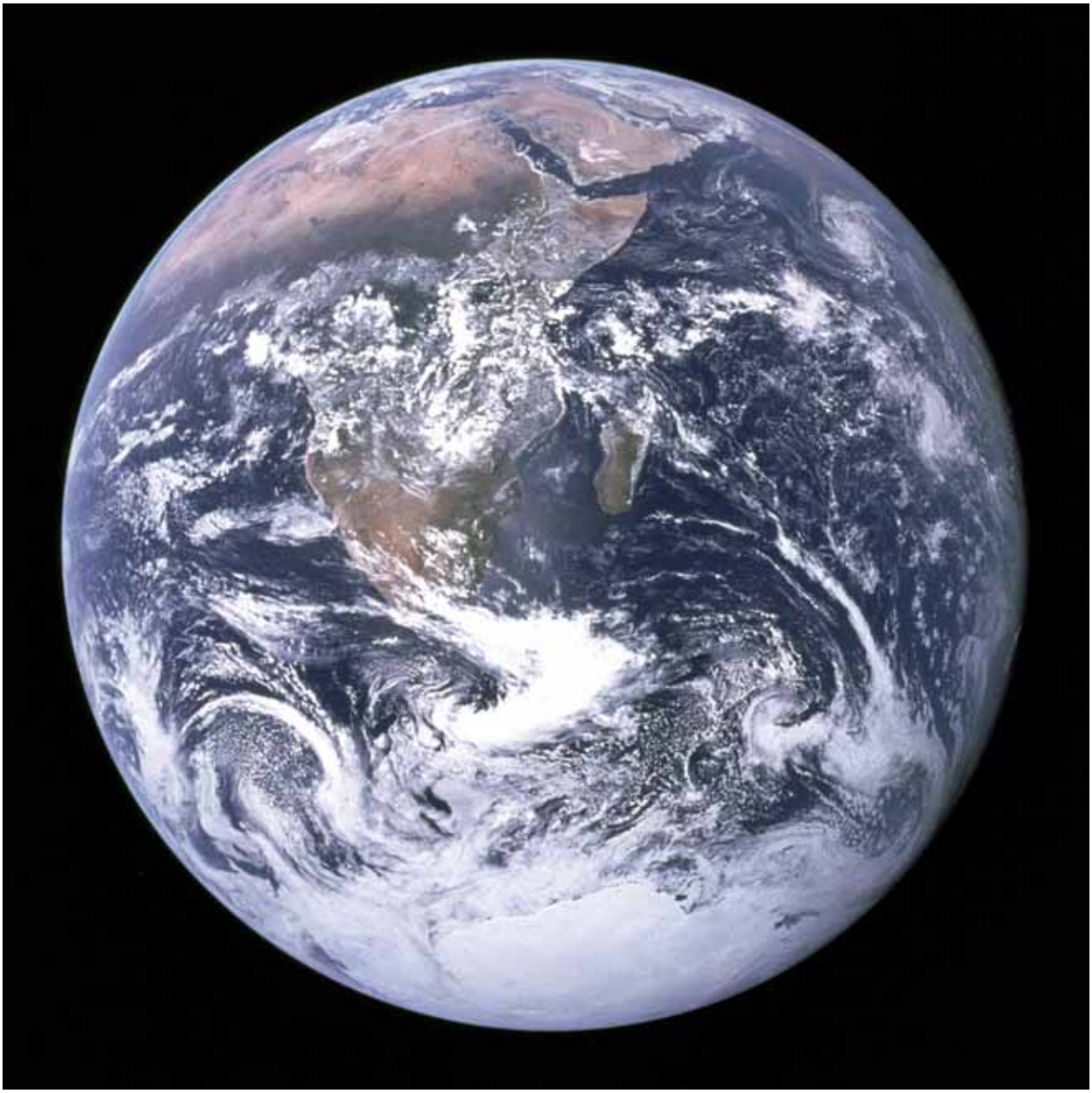

**Figure 4.1 Planet Earth**

A study of polyhedrons, which are spatial figures bounded by polygons, was presented in the previous chapter. Other important spatial forms are studied in this chapter, many of which appear in our everyday lives and are used in the art and architecture worlds.

**4.1 Sphere**

Probably no other spatial figure is so perfect and beautiful as the sphere, which reminds us of our planet Earth. In addition, like the circle, it has esoteric connotations characterizing unity and perfection. A computer generated sphere is shown in Figure 4.2.

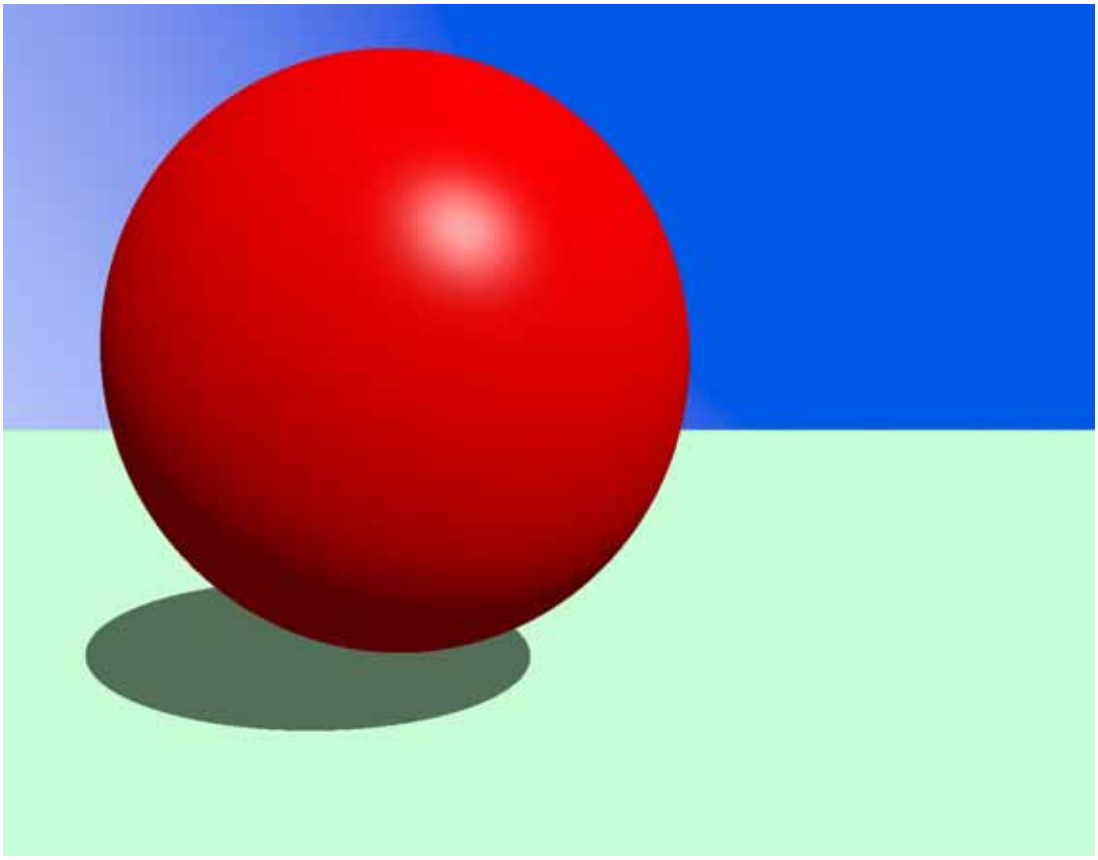

Figure 4.2 The sphere – Hamello - Computer generated

A spherical surface is the set of points on a 3-dimensional surface, lying at the same distance from a central point, called the center of the sphere. The distance from the the center to any point on the sphere is called the radius.

The surface of a ball of clay, for example, is a spherical surface. The entire ball itself is a sphere. There is an analogy between circumference and circle, and between sphere and spherical surface. The circumference has a length while the spherical surface has an area. The circle has an area while the sphere has a volume.

Planet Earth is certainly not a sphere, although its image seen at the start of this chapter resembles one. The term “geoid” is used to designate a body with Earth’s shape. To be more rigorous, one would say that the surface of the Earth is approximately spherical, while the Earth itself is approximately a sphere.

Figure 4.3 shows a body approximately shaped as a sphere, made with LEGOS by Philippe Hurbain. It may seem easy to make it, but the process requires knowing how to put the LEGO pieces in proper order. The instructions for such are given by Philippe Hurbain in his website, indicated in the virtual references given in chapter 12.

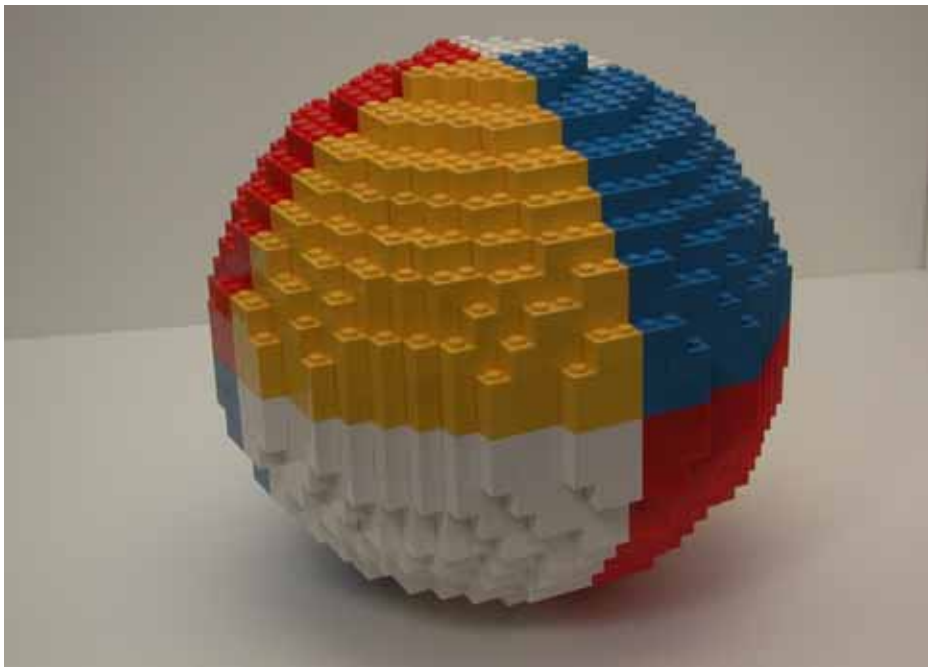

Figure 4.3 LEGOS sphere - Philippe Hurbain

Figure 4.4 shows the amazing spheres of Dick Termes. His peculiar work is studied in detail in chapter 6, which deals with perspectives.

Figure 4.4 Spheres of Dick Termes

Figure 4.5 shows the geometric form known as spherical hubcap. Such a solid is obtained by cutting a sphere with a plane.

Figure 4.5 Spherical hubcaps

The Brazilian National Congress, designed by the award-winning Brazilian architect Oscar Niemeyer (1907-) is a beautiful architectural piece. That magnificent architecture is shown in Figure 4.6. It has two buildings shaped like parallelepipeds, flanked by two others shaped as hubcaps, which are the House of Representatives and the Senate.

Figure 4.6 National Congress in Brasilia, Brazil - Oscar Niemeyer

**4.2 Cone of revolution**

Figure 4.7 shows the form known as cone of revolution. Such a form is seen in every day life for redirecting traffic in roadwork, and in funnels used for pouring liquids into a container.

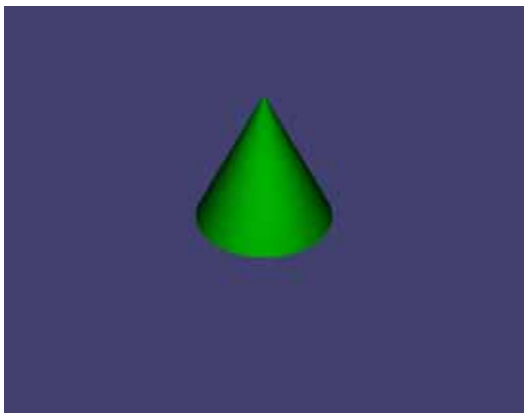

Figure 4.7 Cone of revolution

Two interlaced cones are shown in Figure 4.8.

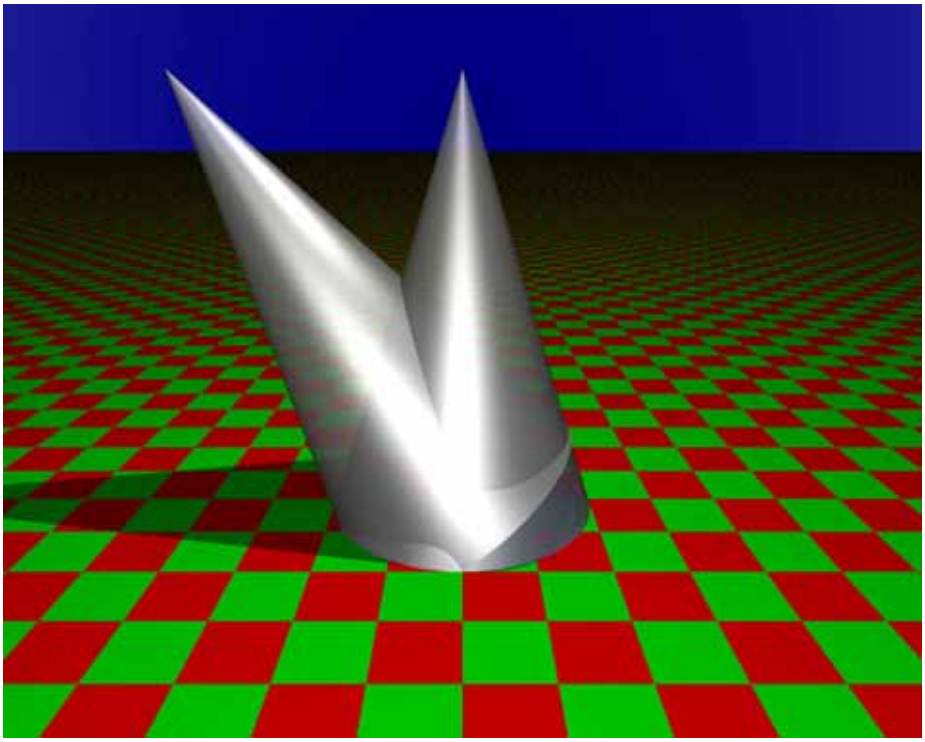

Figure 4.8 Interlaced cones - Hamello - Computer generated

Two interesting figures used in this book for a study of the so-called conics are the double cone and the truncated cone shown in Figure 4.9. Actually, a double cone consists of two cones having a common vertex and the same axis.

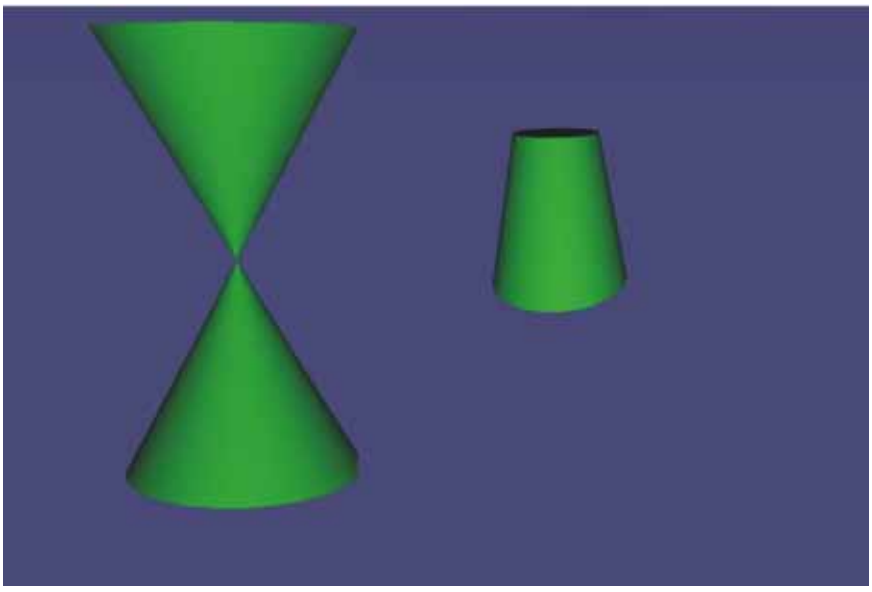

Figure 4.9 Double cone and a truncated cone

We suggest the reader to consult the work of the Czech artist Ivan Kafka, who created several artworks using both cones and truncated cones.

**4.3 Cylinder of revolution**

A cylinder of revolution, shown in Figure 4.10, has two circular bases, and the distance between them is called the height of the cylinder. For the cone and for the cylinder, one can speak, as done with the sphere, of a conic surface and of a cylindrical surface when referring to their periphery, and of a cone and a cylinder when referring to the solids themselves.

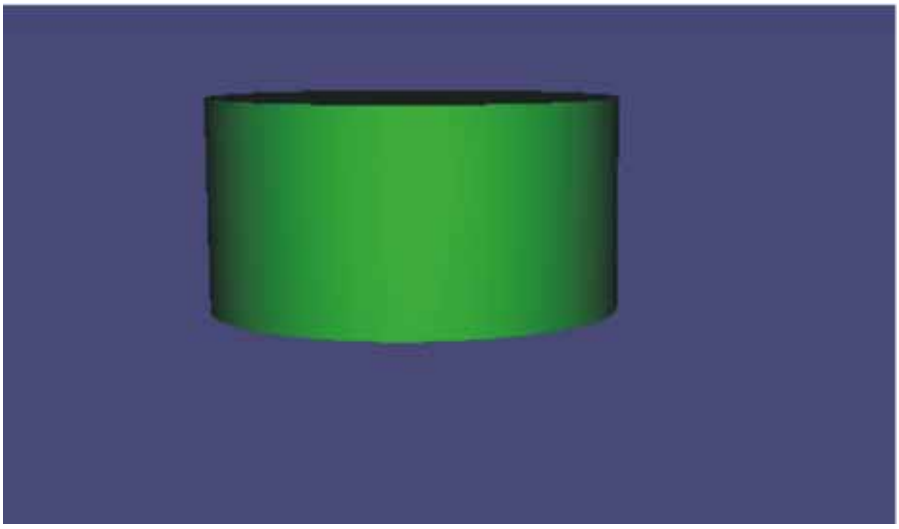

Figure 4.10 Cylinder of revolution

The works of João Galvão, illustrated in Figures 4.11 and 4.12, show how different geometric solids can be used to create beautiful artwork.

Figure 4.11 Untitled - João Galvão

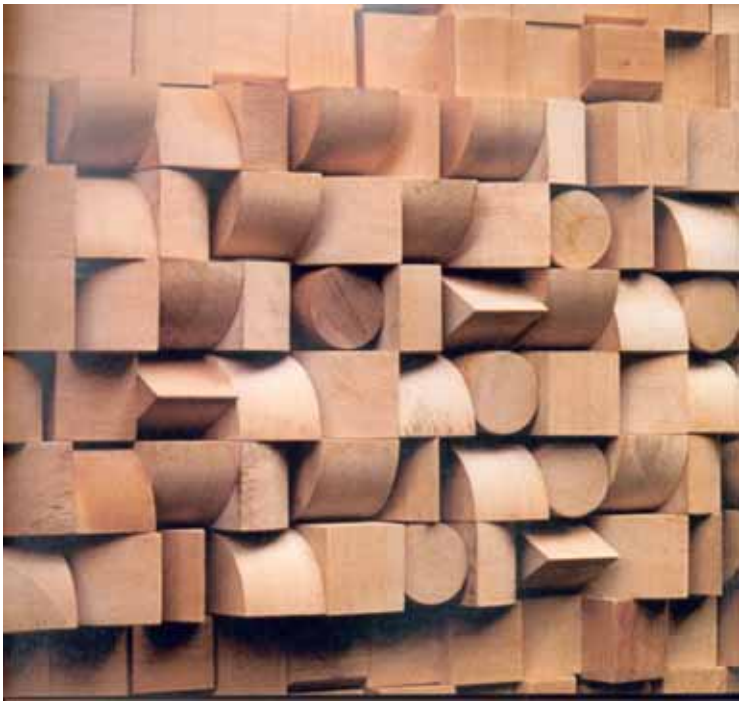

Figure 4.12 Untitled - João Galvão

## 4.4 Conics

### 4.4.1 Generalities

Conics are figures obtained by intersecting a double cone with a plane. Figure 4.13 illustrate the positions a plane may have relative to a double cone.

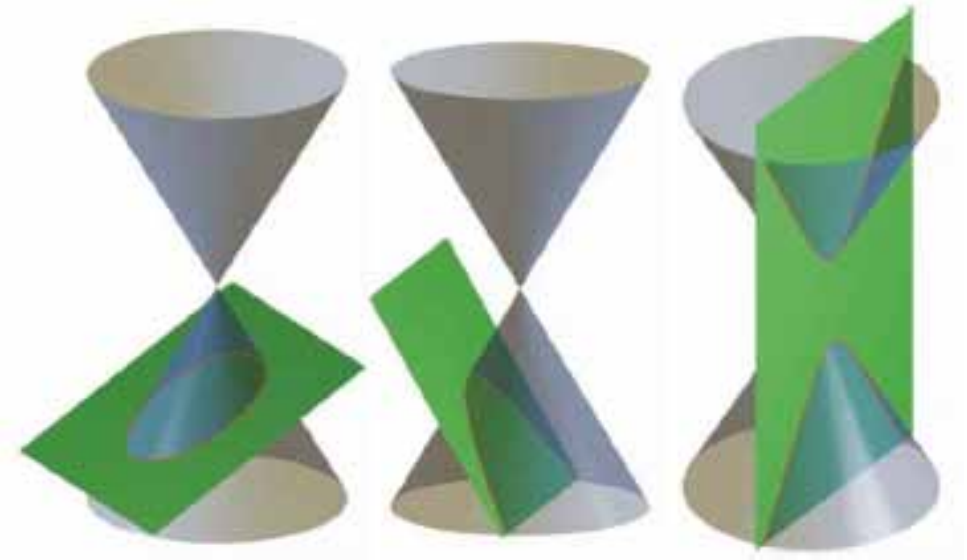

Figure 4.13 Intersection of a plane with the double cone

In the first case, shown at left in the figure, the plane is inclined relative to the axis of the cone and intersects only one of the double cones. In the second case the plane intersects only one of the cones. In the third case the plane is parallel to the axis of the cone and intersects both cones. The curves generated by this process are the ellipse, the parabola, and the hyperbola, respectively. Note that the hyperbola has two branches because the plane cuts both cones.

### 4.4.2 Ellipse and ellipsoid of revolution

Figure 4.14 shows an ellipse and an ellipsoid of revolution. An ellipsoid of revolution is the solid generated by rotating an ellipse around one of its axes. In other words, the ellipse is a planar figure and the ellipsoid of revolution is a solid that reminds us of an egg.

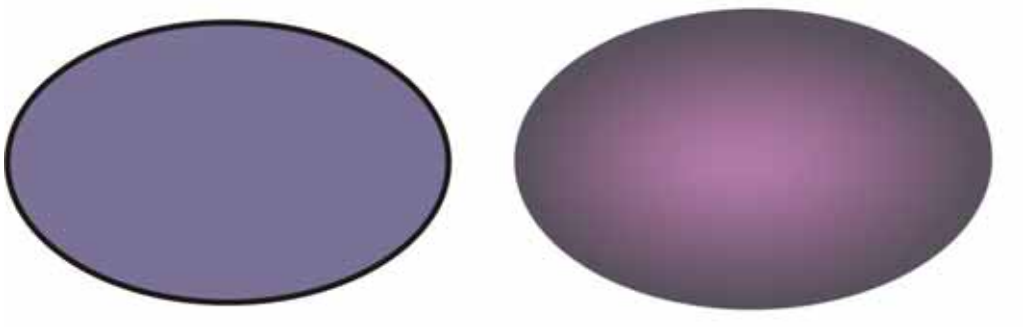

Figure 4.14 Ellipse and ellipsoid of revolution

Figure 4.15 shows a computer generated set of ellipsoids.

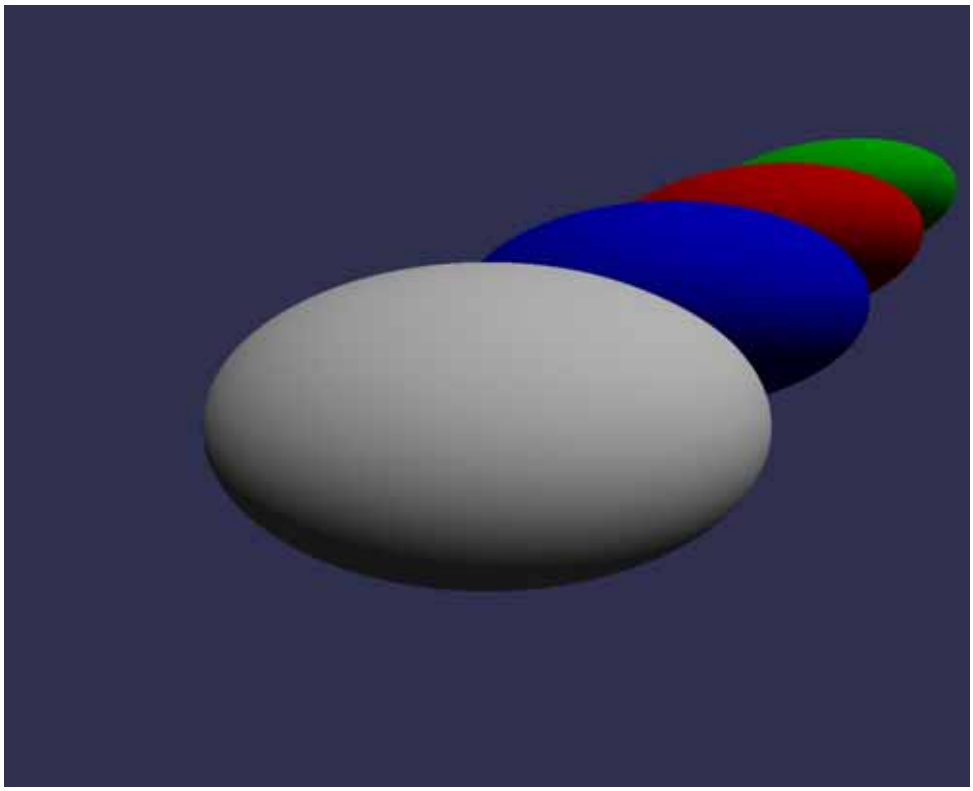

Figure 4.15 Ellipsoids - Hamello - Computer generated

Generally our first contact with the ellipse was when we were studying the solar system and learned that the planets revolve around the Sun in **elliptical** orbits.

The history of science tells us that Claudius Ptolemy (85-165) created the so-called **geocentric** system in which the Earth was fixed and was the center of the universe, with the other celestial bodies revolving around it in circular orbits. Such a theory was accepted as valid for about 14 centuries and was rejected only when Nicolaus Copernicus (1473-1543) proposed the **heliocentric** system, placing the Sun at the center of the universe and the other celestial bodies revolving around it in circular orbits.

Based on his own observations, Galileo Galilei (1564-1642) concluded that the heliocentric system, which he publicly supported, was correct. That created many problems for him with the Church, which rejected any new ideas, especially the one that removed the Earth from the center of the universe. The Inquisition accused Galileo of heresy and forced him to publicly reject his ideas. It is said that he then pronounced the words "**e per si muove**", meaning that the Earth would continue to move in space independently of his recanting. Actually, there is no proof that Galileo said those words.

It was the German astronomer Johannes Kepler (1571-1630), using data collected by Tycho Brahe (1546-1601, who formulated the laws that govern the motion of the planets. Kepler was Tycho Brahe's assistant. Kepler's law states that the planets describe elliptical orbits around the Sun, rather than circular, with the Sun at one of the foci of the ellipse.

The mathematical proof of Kepler's laws is due to Sir Isaac Newton (1642-1727), who is one of the most respected scientists of all times. Newton also formulated the famous law of universal gravitation, which states that bodies attract each other with a force proportional to their masses and inversely proportional to the square of the distances between them.

A composition using ellipses, which reminds the viewer that the circumference is a particular case of the ellipse, is shown in Figure 4.16.

Figure 4.16 Composition with ellipses - Hamello

### 4.4.3 Parabola and paraboloid of revolution

Figure 4.17 shows a parabola and a paraboloid of revolution, the latter being the solid obtained when a parabola is rotated about its axis.

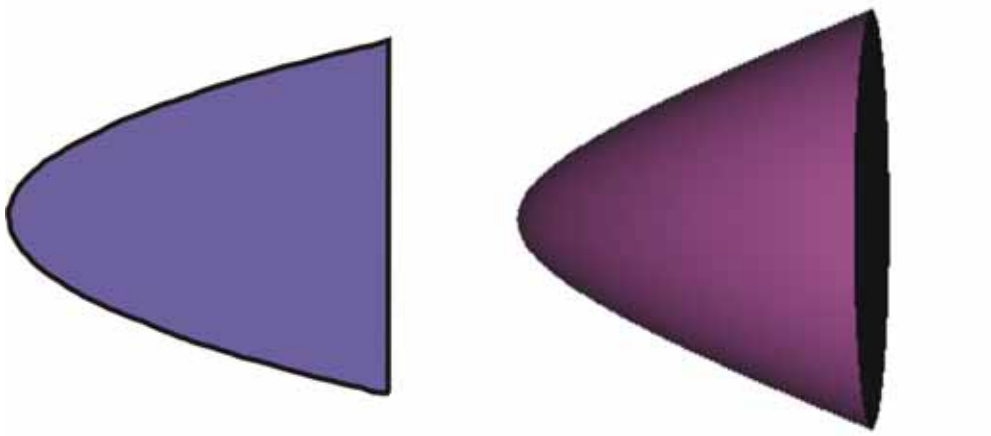

Figure 4.17 Parabola and a paraboloid of revolution

Figure 4.18 shows a computer generated set of paraboloids.

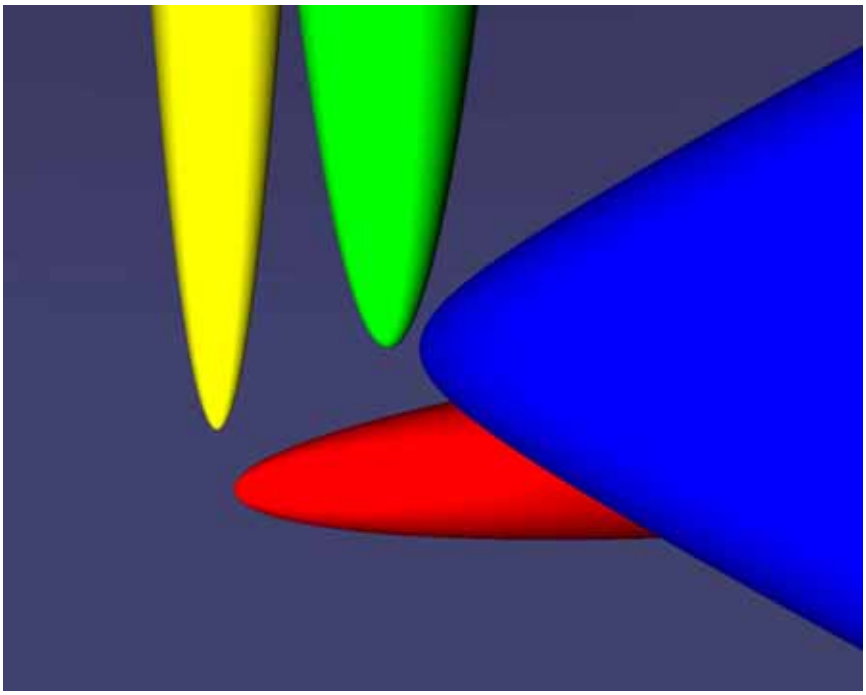

Figure 4.18 Paraboloids - Hamello - Computer generated

The paraboloid has an important property that is used in the well known parabolic antennas. When a beam of energy, such as a signal coming from a satellite, strikes the internal side of a parabolic surface, the incoming rays that are parallel to the paraboloid axis are reflected towards a single point, known as the focus of the paraboloid, concentrating the incoming signal at that point. The idea, then, is to point the axis of the paraboloid in the satellite direction. Since the satellite is at a long distance away, compared to the dimensions of the antenna, the rays arrive at the antenna practically parallel to the axis of the paraboloid, concentrating at its focus where an electronic device is placed to capture the signal.

### 4.4.4 Hyperbola and hyperboloid of revolution

Complementing the study of conics, we show a hyperbola and the hyperboloid of revolution in Figure 4.19. The hyperboloid of revolution is the solid generated by rotating a hyperbola around its axis. Two hyperboloids can be generated in such a process depending on which of the two axes of symmetry shown in Figure 4.19 is used for the rotation. The right part of the figure shows the hyperboloid obtained when the rotation is about the vertical axis shown on the left.

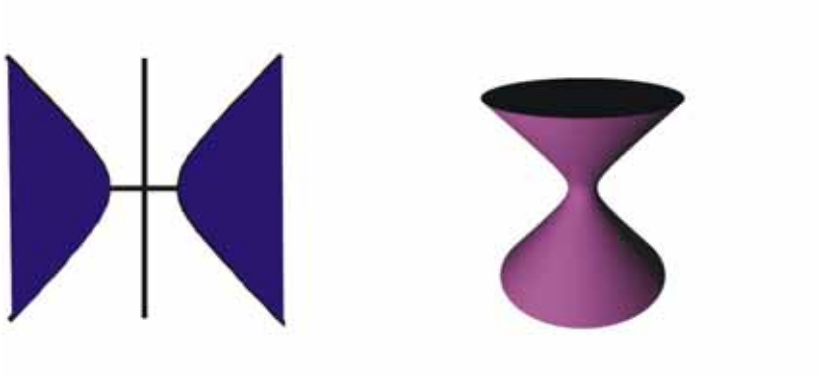

Figure 4.19 Hyperbola and hyperboloid of revolution

Figure 4.20 shows a computer generated work using hyperboloids.

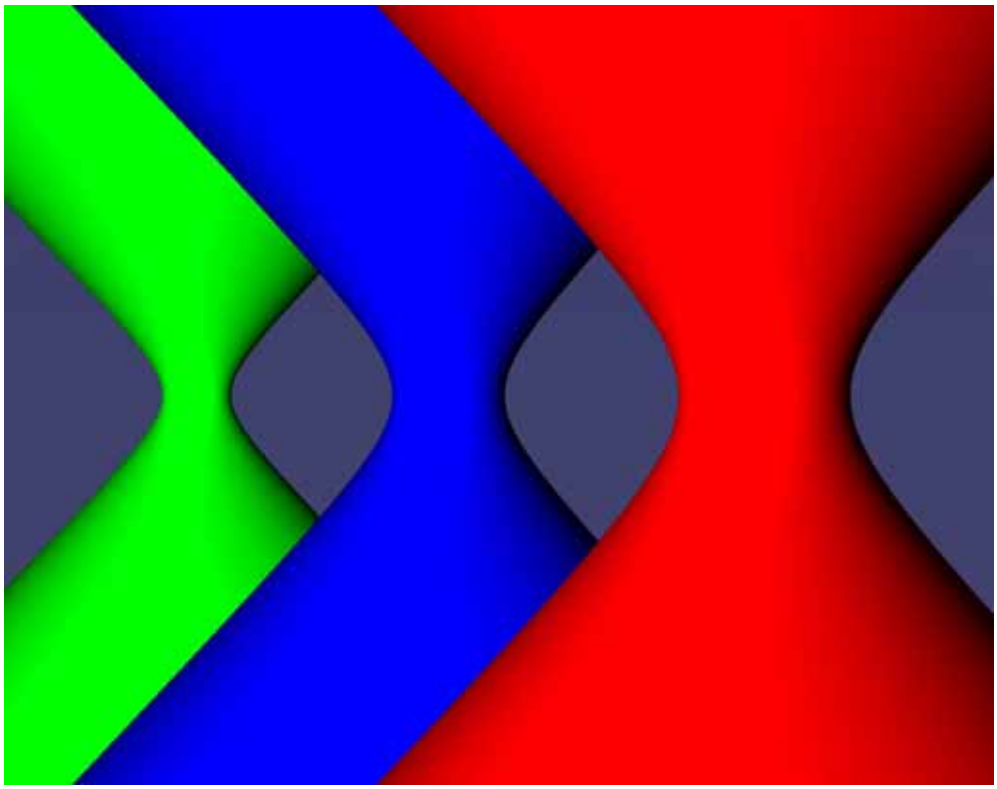

Figure 4.20 Hyperboloids - Hamello - Computer generated

The hyperbolic form is not very common in everyday life, but the reader can view it at home with a simple experiment. Just take a cylindrical lantern, with its axis parallel to a wall, and light it. The figure that appears on the wall, which is the intersection of the light cone with the plane of the wall, is one of the branches of the hyperboloid.

A beautiful example of a hyperbolic structure is the Cathedral in Brasilia, Brazil, designed by the architect Oscar Niemeyer. It is shown in Figure 4.21. Also seen in that figure are four bronze sculptures, designed by the sculptor Alfredo Ceschiatti (1918-1989), representing the Evangelists.

Among other hyperbolic structures, we suggest the interested reader to study the water towers designed by the Russian engineer Vladimir Shukhov (1853-1939), the hyperbolic tower at the port of Kobe, Japan, and the McDonnell planetarium, in St. Louis, Missouri, U.S.A.

Figure 4.21 The Cathedral in Brasilia, Brazil - Oscar Niemeyer

To those who are dedicated to sculpturing, what about making a sculpture in the form shown in Figure 4.22, which resembles a saddle? Such a figure is called a hyperbolic paraboloid, and was not studied in this book.

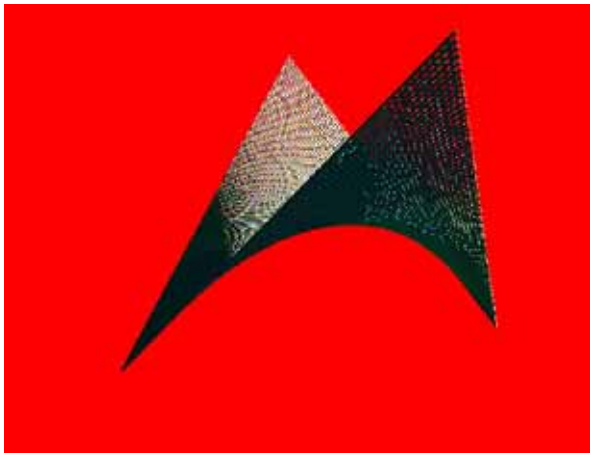

Figure 4.22 Hyperbolic paraboloid

**4.5 The conics and Paul Cézanne**

Several examples of how conics appear in everyday life were presented, and we now end this chapter with an interesting moment in art history associated with Paul Cézanne. Paul Cézanne (1839-1906) was born in Aix-en-Provence, France, and is considered by many to be the "father of modern art". Our particular interest concerns the geometric vision Cézanne had of nature, which became the foundation of modern art. He was introduced to the impressionist group by his friend Camille Pissarro (1830-1903), and participated in the 1874 art exposition of that group. However he had objections to the group's ideas, even believing that their artwork lacked a more formal structure. In fact, Cézanne is generally considered to be part of the post-impressionism group, together with Seurat, Van Gogh, and Gauguin.

An important aspect of Cézanne's work, in addition to its beauty, of course, is the influence he had on Pablo Picasso's Cubism through his planar compositions and his vision of geometry. Picasso even used to say that Cézanne was "his one and only master".

In a letter written in April 1904 to Emile Bernard (1868-1941), a French symbolist painter, Cézanne recommends: " . . . treat nature by means of the cylinder, the sphere, the cone, everything into proper perspective, so that each side of an object or a plane is directed towards a point . . . ".

Figure 4.23 show the three forms mentioned above assembled together. A challenge is to see if one can, following Cézanne's advice, visualize nature "treated" by these forms.

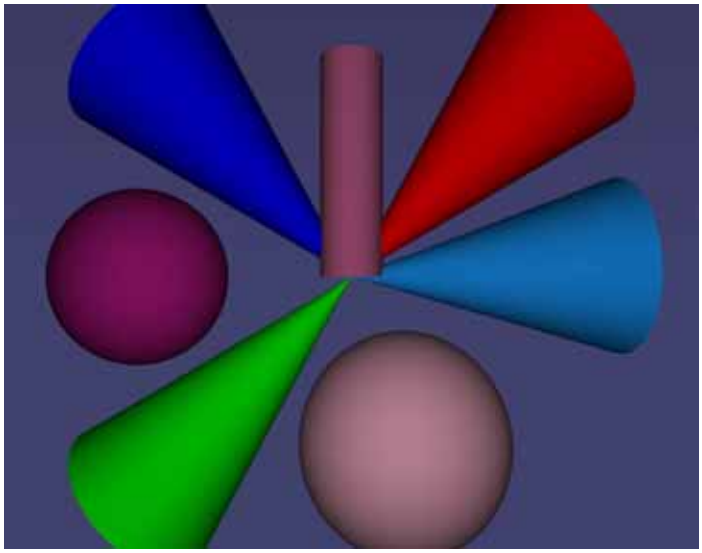

Figure 4.23 Cone, sphere, and cylinder - Hamello - Computer generated

## 4.6 Helices and helicoides

### 4.6.1 Helices

A helix is a three dimensional curve that lies either on a cylindrical or on a conic surface. We then have either a cylindrical or a conical helix, as shown in Figure 4.24.

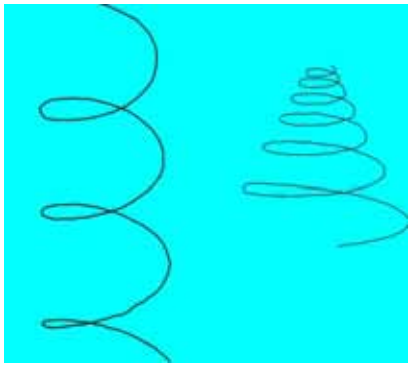

Figure 4.24 Cylindrical and conical helices

Springs, which are present in our everyday life, are good examples of helices. And, speaking of springs, who does not remember that long cylindrical spring toy called Slinky (see Figure 4.25), with its interesting properties such as being able to travel down a flight of stairs step by step?

Figure 4.25 The slinky

### 4.6.2 Helicoids

A helicoid is a tri-dimensional surface generated by a helix, as shown in Figure 4.26

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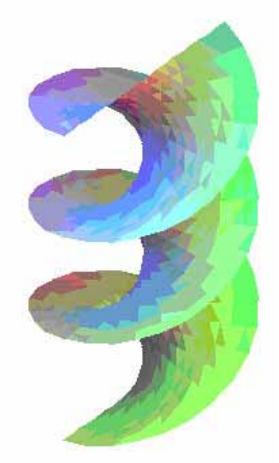

Figure 4.26 Helicoid

We end this chapter by presenting some of the wonderful sculptures of Ascânio Maria Martins Monteiro, normally known as Ascânio MMM. He was born in Portugal in 1941 and immigrated to Brazil, settling in Rio de Janeiro in 1959. Many of his sculptures are in public places in Brazil and in other countries. Figures 4.27 to 4.29 show three of his beautiful sculptures.

Figure 4.27 "Módulo6.5" - Ascânio MMM - "Centro Administrativo São Sebastião", Rio de Janeiro, Brazil

Figure 4.28 "Módulo 8.4" - Ascânio MMM - "Centro Empresarial Rio" in the Botafogo beach area of Rio de Janeiro, Brazil

Figure 4.29 “Módulo 1.3” - Ascânio MMM - Daniel Maclise building in the “Cosme Velho” neighborhood, Rio de Janeiro, Brazil

# 5

## Composing a canvas with polygons: Tessellations

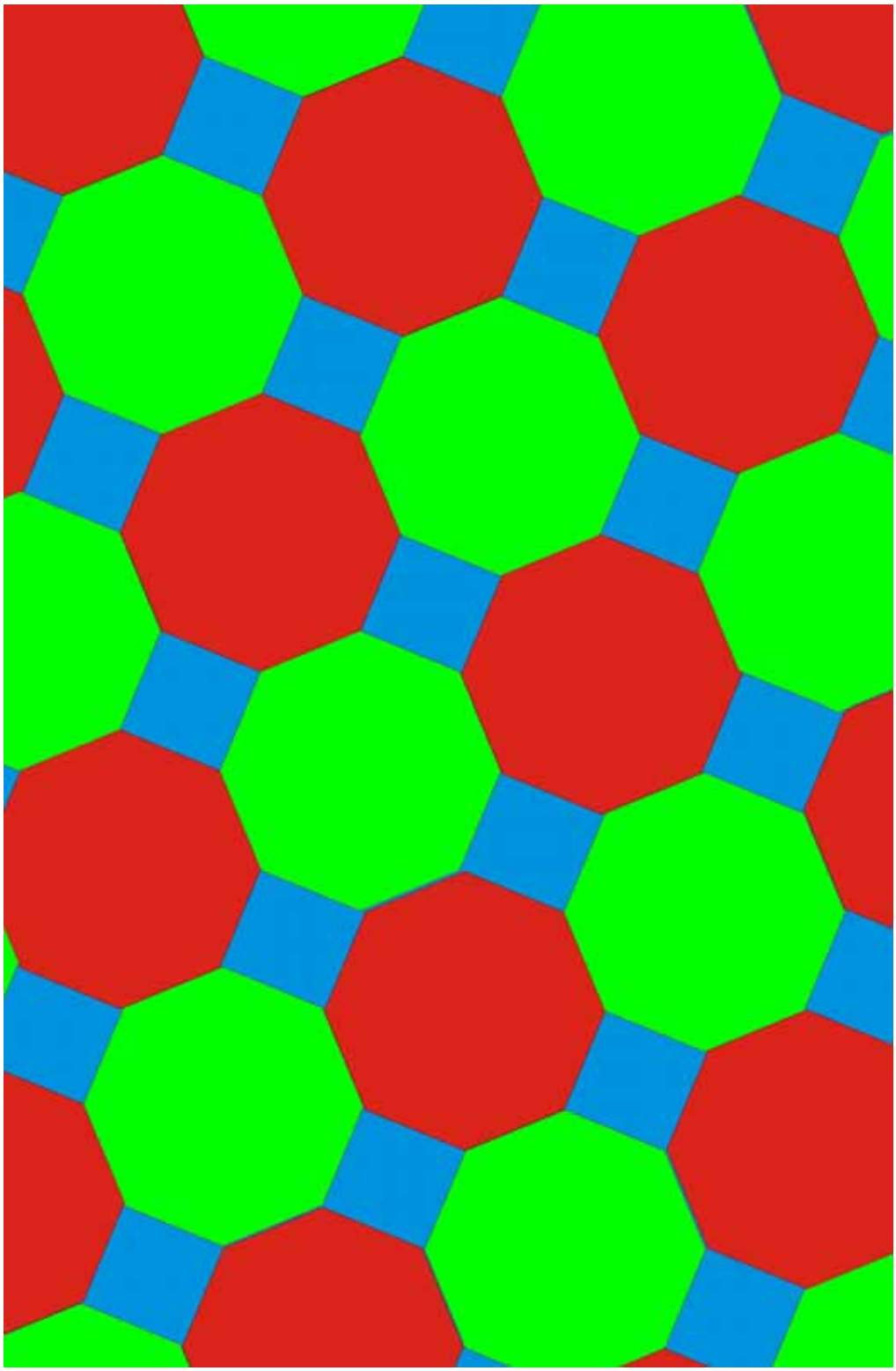

**Figure 5.1 Octagons and squares**

## 5.1 General concepts

Suppose we want to create a geometric painting by filling the entire canvas with polygons, without any gap or overlap, as illustrated in Figure 5.1 with octagons and squares. Is there any theory or any rule that could help us to do so?

By examining the surface of a tiled kitchen or bathroom, for example, one sees an entire area filled with juxtaposed small blocks, which are normally squares or rectangles, with no empty space left. Such a process is called tessellation; the word tiling is also commonly used.

Why are tiles normally shaped as either squares or rectangles? The answer is simple: it is because we can cover an entire surface, leaving no gaps, by placing squares side by side in all four directions.

Would it be possible to cover an entire rectangular plane with tiles of different shapes? Let us analyze such a possibility.

## 5.2 Tessellations with regular polygons

Since a curve surrounding a point sweeps 360 degrees, juxtaposition of polygons around a common vertex leaving no gaps between them implies that the sum of the internal angles of the polygons must be 360 degrees. All possible patterns for filling a plane using only regular polygons, leaving no gaps between them, are shown in Figure 5.2.

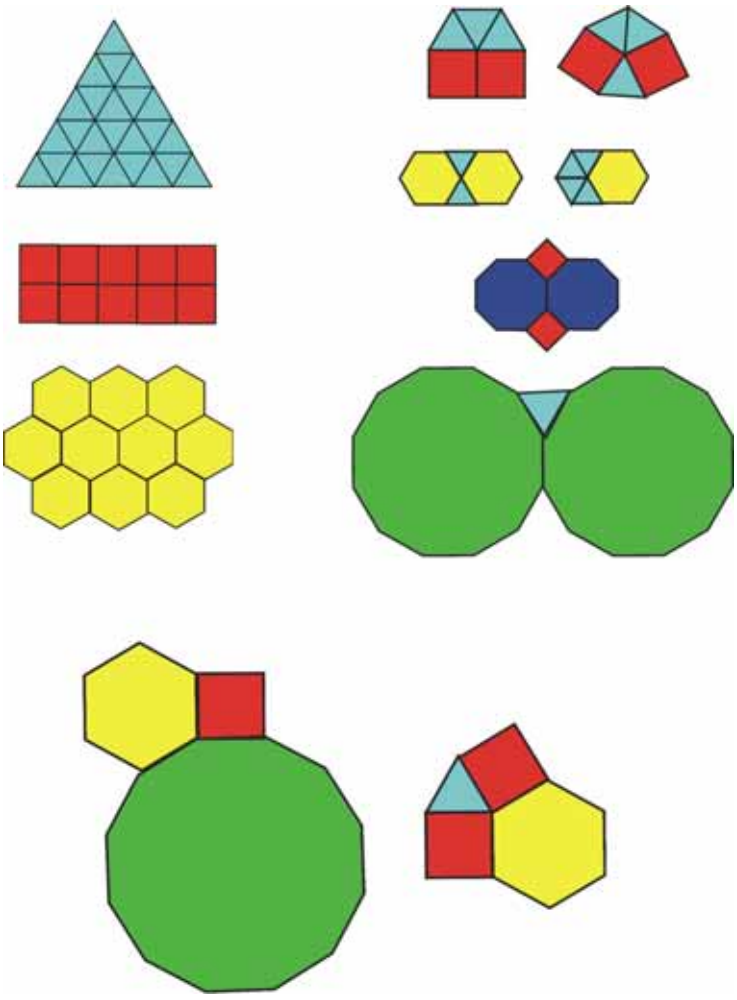

Figure 5.2 Possible tessellations with regular polygons

As seen in Figure 5.2, choosing a single type of regular polygons, i.e., one of the polygons with equal sides, to make a tessellation, only the square, the equilateral triangle, and the hexagon allow a plane to be filled without any gaps because the interior angles of those polygons are 60, 90, and 120 degrees, respectively. For the same reason, the use of two types of regular polygons allows a plane to be filled in six different ways, which are:

| Composition | Possibilities |
|---|---|
| Squares and equilateral triangles | 2 |
| Hexagons and equilateral triangles | 2 |
| Octagons and squares | 1 |
| Dodecagons and equilateral triangles | 1 |

Finally, there are only two possibilities, shown in the following table, if three types of regular polygons are used.

| Composition | Possibilities |
|---|---|
| Hexagons, squares, and equilateral triangles | 1 |
| Dodecagons, hexagons, and squares | 1 |

In summary, there are a total of 11 possible tessellations if only regular polygons are used, namely: three using only one type of polygon, six if two types are used, and two if three types are used.

Figure 5.3 shows a composition using hexagons, squares, and equilateral triangles, and Figure 5.4 shows a composition using hexagons and triangles.

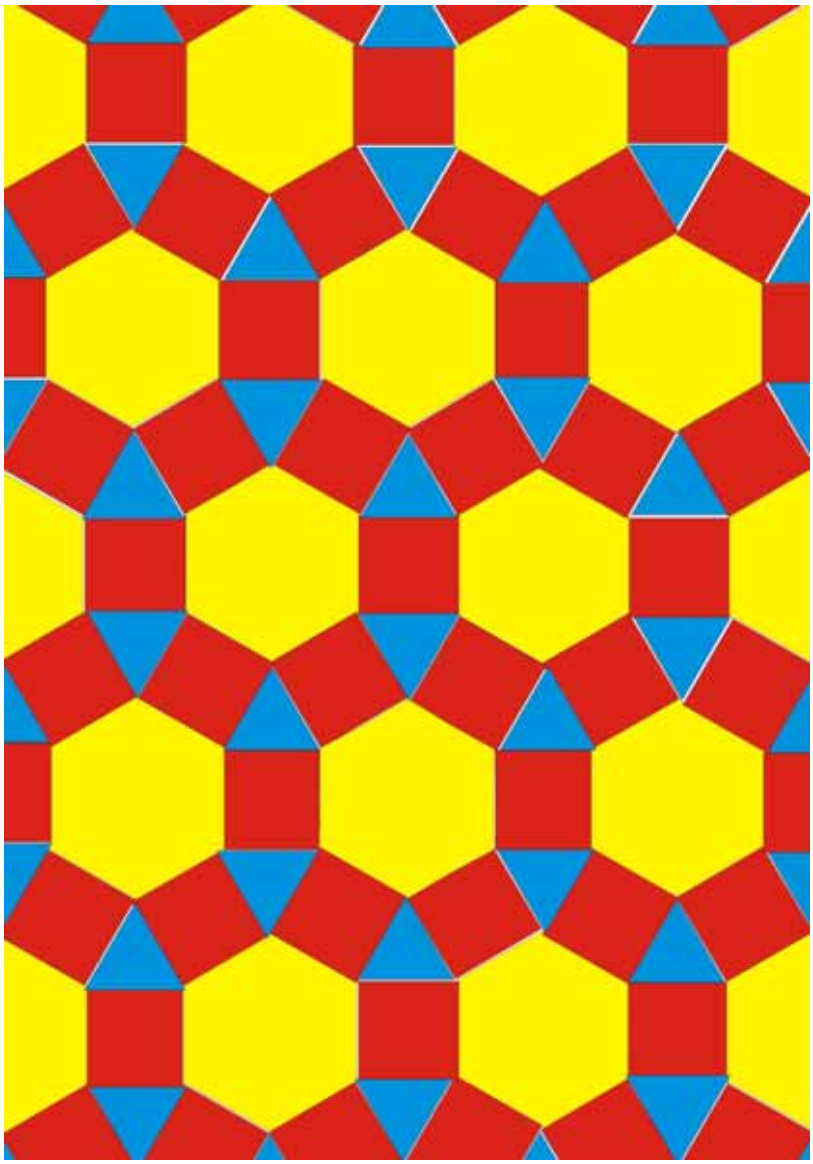

Figure 5.3 Composition using hexagons, squares, and triangles

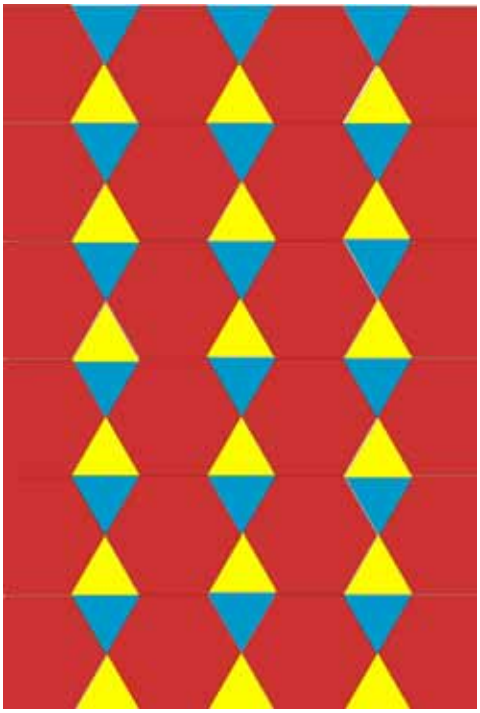

Figure 5.4 Composition using hexagons and triangles

**5.3 Tessellations with irregular polygons**

Up to this point our study has been restricted to using regular polygons. If it is desired to use irregular polygons to fill a plane without gaps, one needs only to adjust the shape and the dimension of the pieces. Figure 5.5 shows a composition with irregular polygons completely filling a canvas, which is, then, a tessellation. It is interesting to notice the illusion of depth in some parts of the planar image shown in Figure 5.5. This is due to the different colors used.

Figure 5.5 Tessellation with irregular polygons

A multicolored tessellation reminds us of a patchwork, where assorted pieces of fabric material are sewn together to make a bedspread or a tapestry, for example.

At this point, it is suggested the reader make a tessellation and analyze the resulting image in detail.

**5.4 Tessellations and nature**

The tessellations presented so far were geometrically created, but they can also be seen in nature. Figure 5.6 shows the planar projection (a photograph) of a beehive, consisting in a juxtaposition of hexagonal cells completely filling a plane in a perfect and efficient manner, without wasting any space.

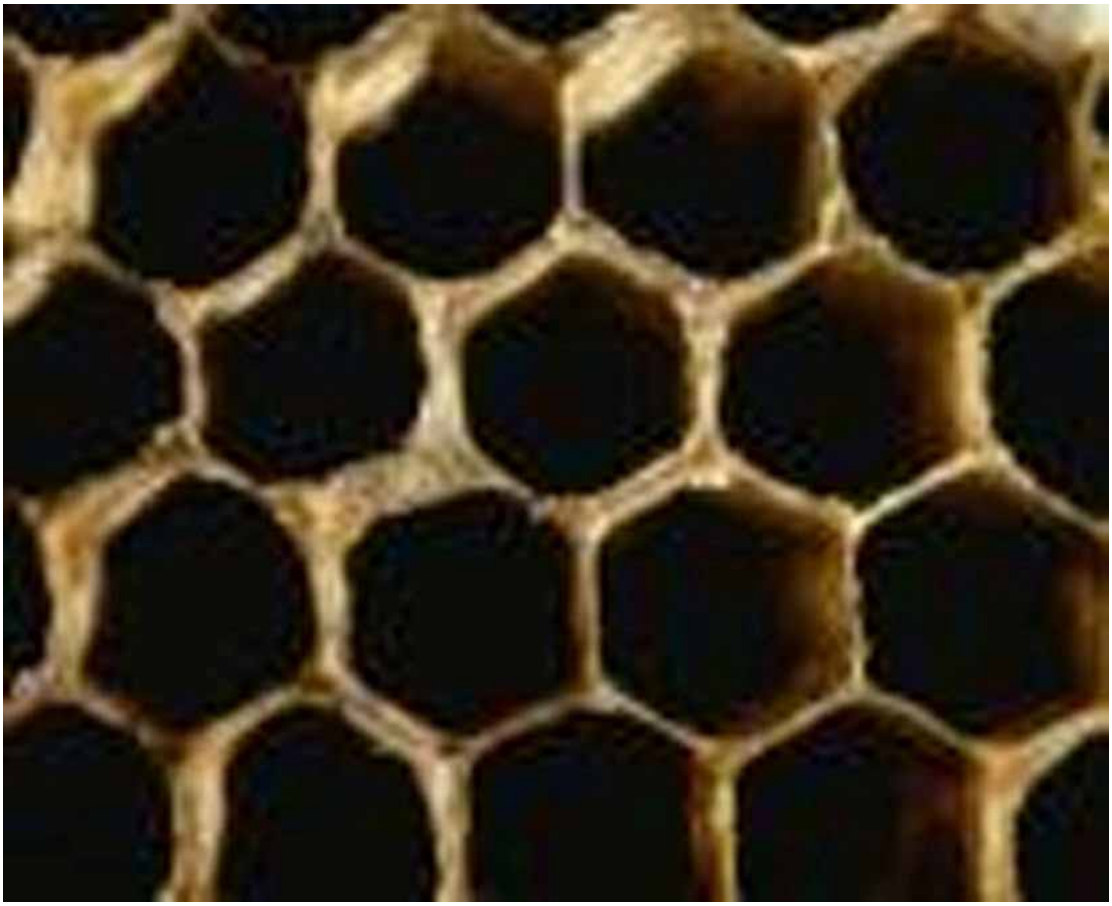

Figure 5.6 Beehive

Another interesting, but sad, example of a natural tessellation is seen in some regions where the Earth surface becomes so dry to the point of breaking up and forming a random pattern. Such patterns are random tessellations.

From a mathematical viewpoint, tessellations, or tilings, may be classified as periodic and non-periodic. A tessellation is periodic when a pattern is repeated when composing a plane (as, for example, in the case of a beehive, where the plane is composed by hexagons), and non-periodic when there is no pattern repetition.

**5.5 Penrose tessellations (or tilings)**

There is a group of well known non-periodic tessellations discovered by the mathematician and physicist Roger Penrose (1931-). Besides their beauty, they played an important role in crystallography studies, especially in the so-called “quasicrystals”. A fascinating fact is that they can be generated using the two types of parallelograms shown in Figure 5.7: one, in blue, with internal angles equal to 36 and 144 degrees, and the other, in red, with internal angles equal to 108 and 72 degrees.

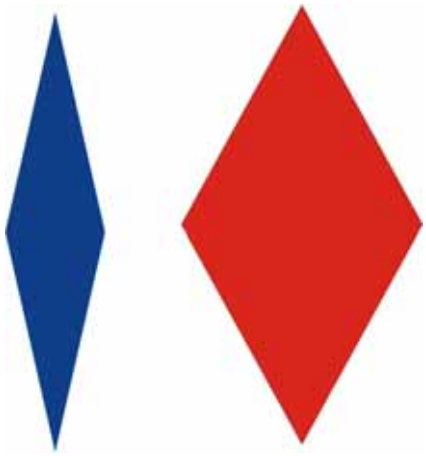

Figure 5.7 The two basic pieces of a Penrose tessellation

Figure 5.8 shows one of Penrose's tessellations, with the same colors shown in Figure 5.7 for easy identification.

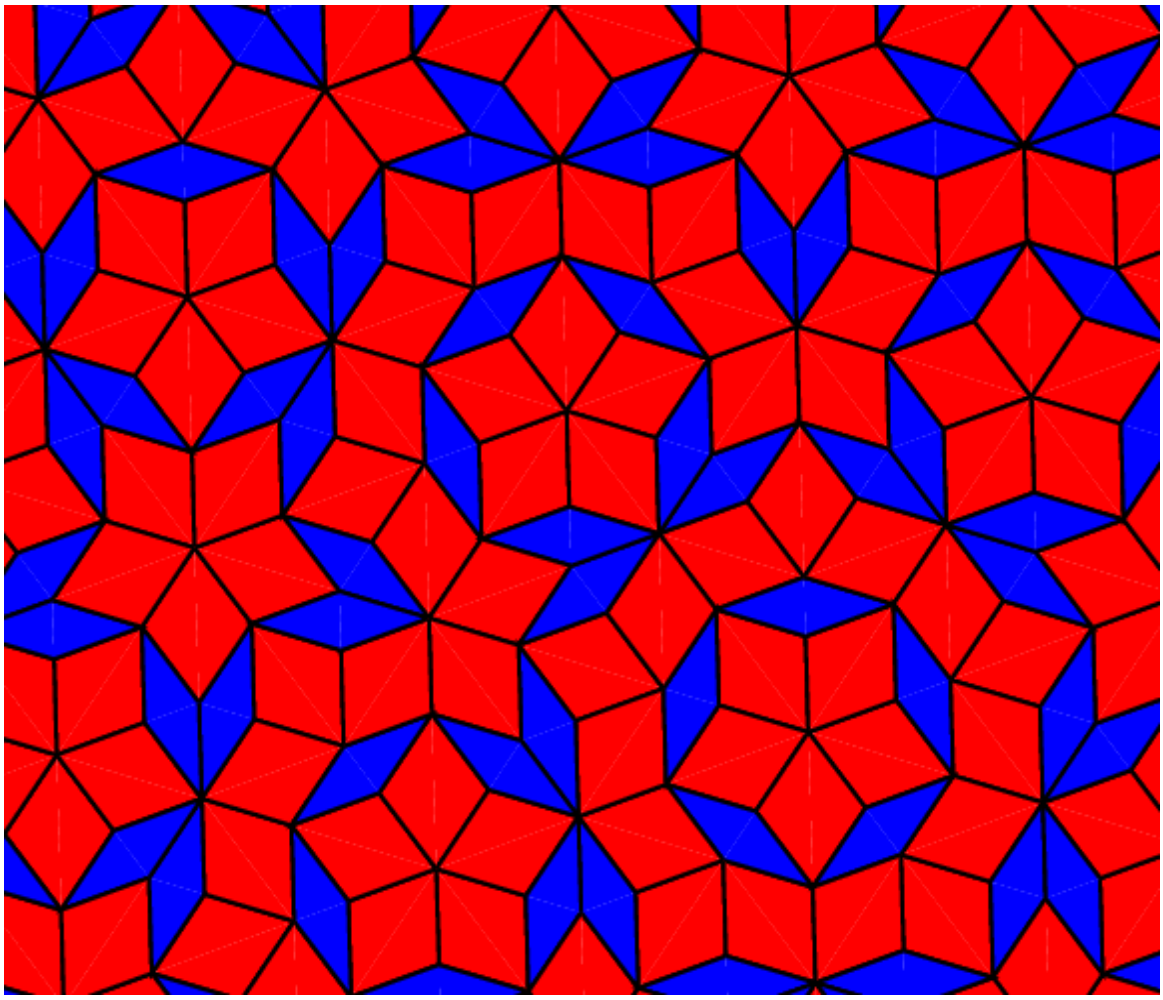

Figure 5.8 An example of a Roger Penrose tessellation

Since two geometric figures were used for composing the tessellation shown in Figure 5.8, one might be tempted to think that only symmetric and periodic tessellations can be made using them. Actually, in a periodic tessellation a pattern is repeated indefinitely, but this is not the case for Penrose tessellations. A first glance of Figure 5.8 may give the impression that a small portion of it is repeated indefinitely. However, a more careful look discloses that the figure is not periodic, which is more easily seen by looking at it from a larger distance.

Penrose tessellations are the subject of fascinating mathematical studies. The reader who is interested in the subject matter can find a wealth of information either by searching the Internet using the keywords “Penrose tilings” or by consulting a book on the subject matter. There are even computer programs for generating Penrose tessellations, some of which are listed in chapter 12.

**5.6 Final considerations**

Our brief study in this chapter dealt only with tessellations made with polygons, which might leave the reader thinking that a plane can be filled without gaps only if polygons are used for such. Actually, it is possible to compose a plane with other patterns, but such a study is beyond the scope of this book. A composition made with irregular pieces arranged in a manner that fills the entire plane is shown in Figure 5.9 only to illustrate such a possibility. Such tessellations are known as Heesch Tessellations, in honor of the German mathematician Heinrich Heesch (1906-1995) who did research on tilings for a brief period of his impressive mathematical career.

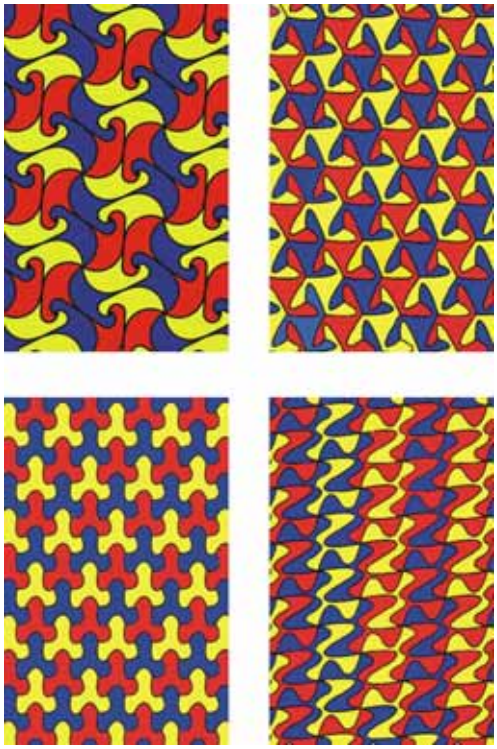

Figure 5.9 Examples of Heesch tessellations

We cannot end this chapter without mentioning the work of the Dutch graphic artist Maurits Cornelis Escher (1898-1972), known as M. C. Escher. Extensive information about Escher and his work is available on the Internet, and his website is listed in Chapter 12.

# 6

## Perspective

**Figure 6.1 “Atom and Eve”- Dick Termes**

## 6.1 Introduction

A drawing on a flat piece of paper, or a painting on canvas, of a scene containing objects in three-dimensional space is something we all have seen. Figure 6.2 shows one of such scenes. Although the drawing is on a flat surface, one is able to perceive how the blocks are distributed in space. A distinguishing characteristic of the drawing, known since antiquity, is the sensation that the buildings seem to become smaller as they are farther and farther away.

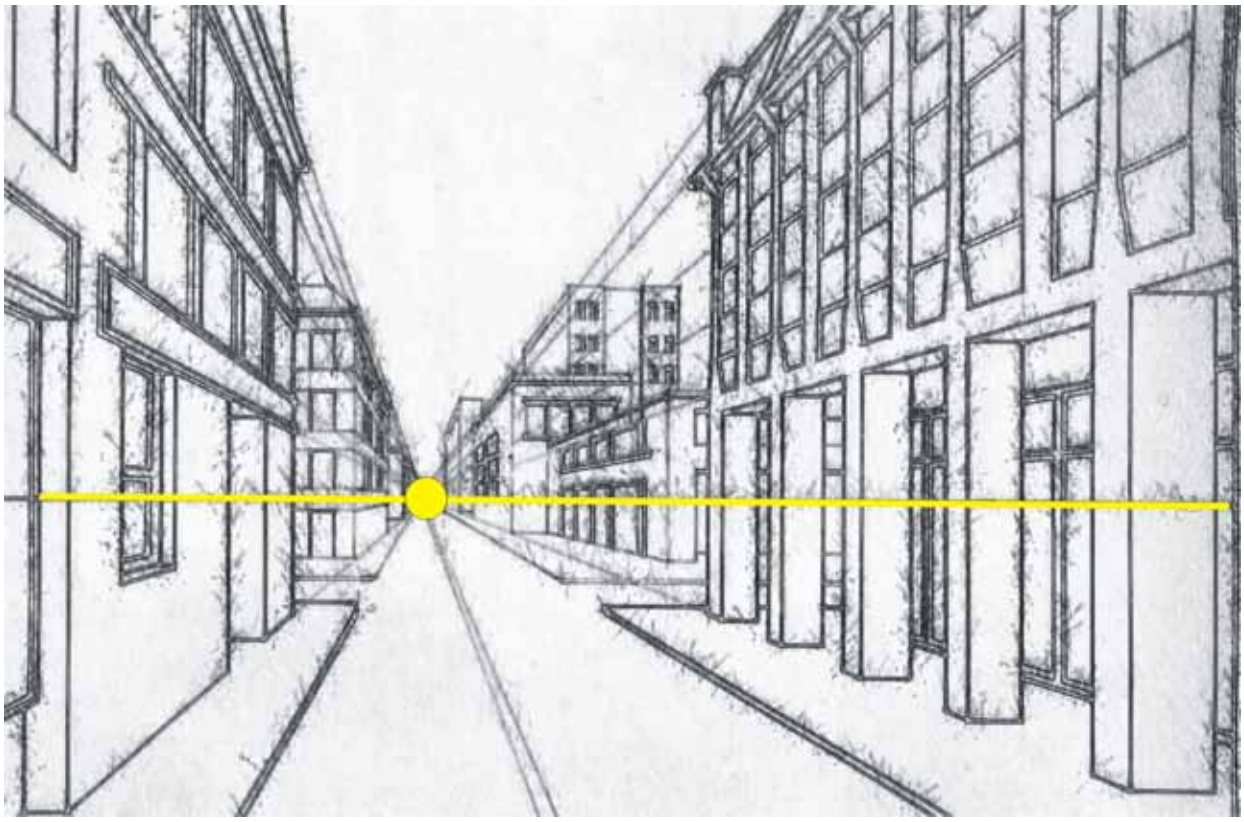

Figure 6.2 A three dimensional scenery

We inserted in Figure 6.2, as a reference, a yellow horizontal line and a yellow point on that line where the image converges. These are important elements, and we will discuss them later.

There have been since earlier times several attempts to represent three-dimensional space on a flat surface, and artists from several cultures contributed to the evolution of the techniques for doing so. The geometric technique for representing three-dimensional space on a flat surface is called Linear Perspective.

It was only in the 13$^{th}$ and 14$^{th}$ centuries that artists, such as Giotto di Bondone (1267-1337) and Cimabue (circa1240-1302), among others, used more established techniques to represent three-dimensional space, although those first attempts were not based on formal grounds.

Figure 6.3 shows a Fresco painted by Giotto in 1305, “Christ before Caiaphas”, revealing his attention to the placement of the characters in the painting. In the foreground are Jesus and the steps that turn the viewer’s attention to the far right where Caiaphas, the high priest and president of the Sanhedrin, is seated on his throne. Some notions of perspective can already be seen in that painting, although it is not yet a formally structured “space”.

The invention of the method called Linear Perspective is credited to the Italian sculptor and architect Filippo Di Ser Brunellesco (1377-1446), whose important creations include the dome of the Cathedral of Florence. Unfortunately there are no direct official records of Brunelleschi’s work, but it is believed that his

method was passed on to other artists such as Masaccio (1401-1428), Masolino (1383-1447), and Donatello (1386-1466).

Figure 6.3 Christ before Caiaphas - Giotto - Fresco - Scrovegni Chapel, Padua, Italy

One of the oldest works using more defined notions of perspective is the famous golden bronze relief "Feast of Herod", shown in Figure 6.4, created between 1425 and 1427 by Donato di Niccolò di Betto Bardi, better known as Donatello. That work depicts the moment when the head of Saint John is presented to King Herod at the request of Princess Salome. Notice the horror expressed by the guests and by King Herod himself.

Figure 6.4 Feast of Herod - Donatello - Siena Cathedral, Italia

**6.2 After all, what is a linear perspective?**

Figure 6.5 shows in different colors all the parallel sides of a parallelepiped: the horizontal ones in blue, the vertical ones in green, and the ones in red apparently penetrating the page.

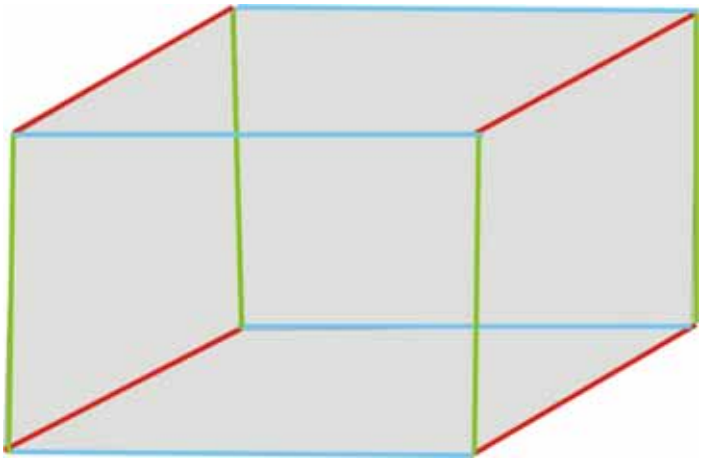

Figure 6.5 Parallel lines of a parallelepiped

The idea behind a linear perspective is to represent parallel lines as lines converging towards conveniently chosen points, called **vanishing points**, located on an imaginary line, called the **horizon line**, drawn at eye level.

There are several types of perspective, and they are classified according to the number of vanishing points used: perspective with one vanishing point, with two vanishing points, and so on. The most common are the ones with one vanishing point, widely used during the Renaissance, and the ones with two vanishing points, which are widely used in architecture. Perspectives with additional vanishing points are also shown in this chapter since some beautiful artwork using them are presented later. Returning our attention to Donatello's work shown in Figure 6.4, notice the vanishing point near the shoulder of the musician at the window in the middle of the figure.

Figure 6.6 shows the parallelepiped of Figure 6.5 drawn in perspective. The horizon line and the single vanishing point (point VP), to which the parallel red lines converge, are also shown.

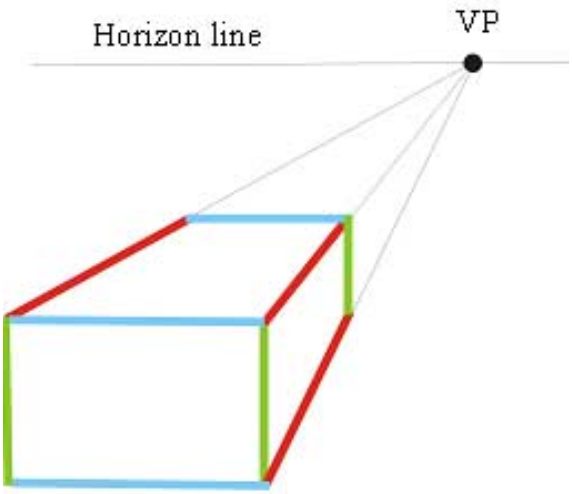


Figure 6.6 Parallelepiped in a perspective with one vanishing point

Several parallelepipeds also drawn in a perspective with one vanishing point are shown in Figure 6.7, with the horizon line drawn at the observer's eye level. Notice that the two figures below the horizon line are seen from above it, and the block above that same line is seen form underneath it.

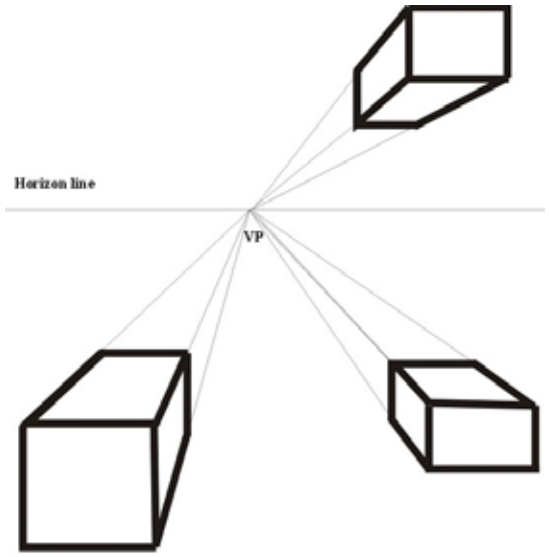


Figure 6.7 Perspective with one vanishing point

A landmark work that illustrates the use of a perspective with one vanishing point is the 1482 Sistine Chapel Fresco by the Italian painter Pietro Perudino (1446-1524). That painting, known as "The Delivery of the Keys" and shown in Figure 6.8, represents the first Pope, Saint Peter, receiving the keys to heaven from the hands of Christ himself. Notice the clever use of perspective, with the vanishing point located approximately in the middle of the entrance door essentially drawing the viewer's attention to that part of the painting.

Figure 6.8 Delivery of the Keys - Pietro Perudino - Sistine Chapel, Rome, Italy

Figure 6.9 shows another famous painting, this one by Raffaello Santi or Sanzio (1483-1520), better known as Raphael, literally called the "The School of Athens" and probably painted between 1509 and 1510. It is also a perspective with one vanishing point, which is approximately located halfway between the two persons in the middle of the painting. The painting is also remarkable because the people depicted on it are important personalities. The two persons at the center are Plato, holding his book "Timaeus" and with his finger pointing skyward, and Aristotle (384 BC-322 BC), holding his book "Ethics" with his hand pointing downwards. Pithagoras, writing a book, and Euclides surrounded by his students, are also represented in the painting.

Figure 6.9 The School of Athens - Raphael - Apostolic Palace, Vatican

Since one's attention tends to turn towards the vanishing point, the renaissance artists placed important elements of their painting near that point. Thus, it is not by chance that Plato and Aristotle, the two most important people in the painting, were placed near that point.

**6.3 Perspective with two vanishing points**

In a perspective with two vanishing points one set of parallel lines of a parallelepiped is represented as lines converging to one point, and another set as lines converging to another point. Again, the convergence points are called vanishing points, as the ones labeled VP1 and VP2 in Figure 6.10.

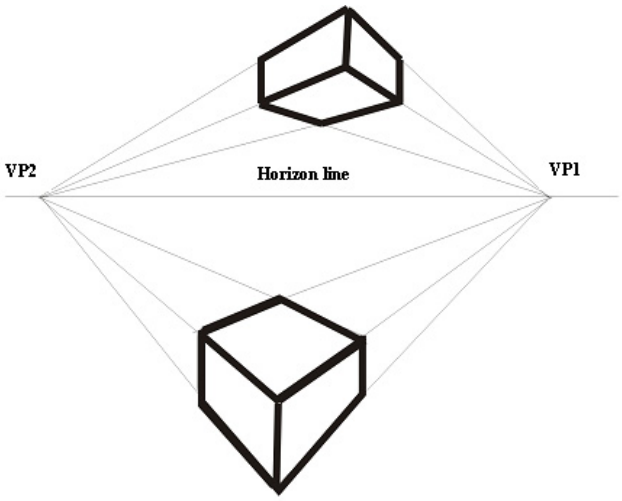


Figure 6.10 Perspective with two vanishing points

What the artist needs to do to draw a perspective is to sketch his work by placing the objects in the areas of interest, and then draw the parallels as lines converging to the vanishing points. By proceeding in this manner, more distant objects will automatically appear smaller.

The location of the horizon line and of the vanishing points has a significant effect on the finished work. We suggest the reader redo the examples given previously by using different locations for the vanishing points in order to see the resulting effect they have in the perspective.

Perspectives with two vanishing points are widely used in Architecture and in Engineering. Figure 6.11 shows a building block using such a perspective.

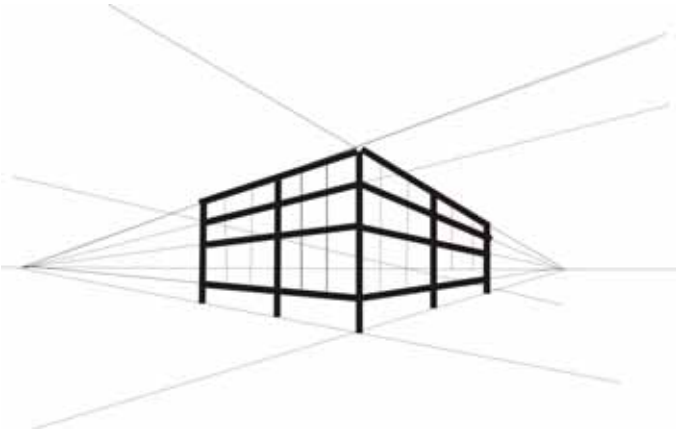

Figure 6.11 Sketch of a building block with two vanishing points

**6.4 Perspective with a larger number of vanishing points**

Perspectives with one and two vanishing points are the best-known ones, but more vanishing points can certainly be used. As those perspectives are an extension of the concepts associated with perspectives with one and two vanishing points, there are some rules that are generally followed when drawing them, The reader interested in pursuing this further can find the details in Dick Termes' book "New Perspective Systems", listed in Chapter 11. The use of a large number of vanishing points allows one to visualize an entire space around a point, yielding interesting images such as those created by Dick Termes. This is illustrated in Figure 6.12, showing parallelepipeds in perspectives with 3, 4, and 5 vanishing points, respectively.

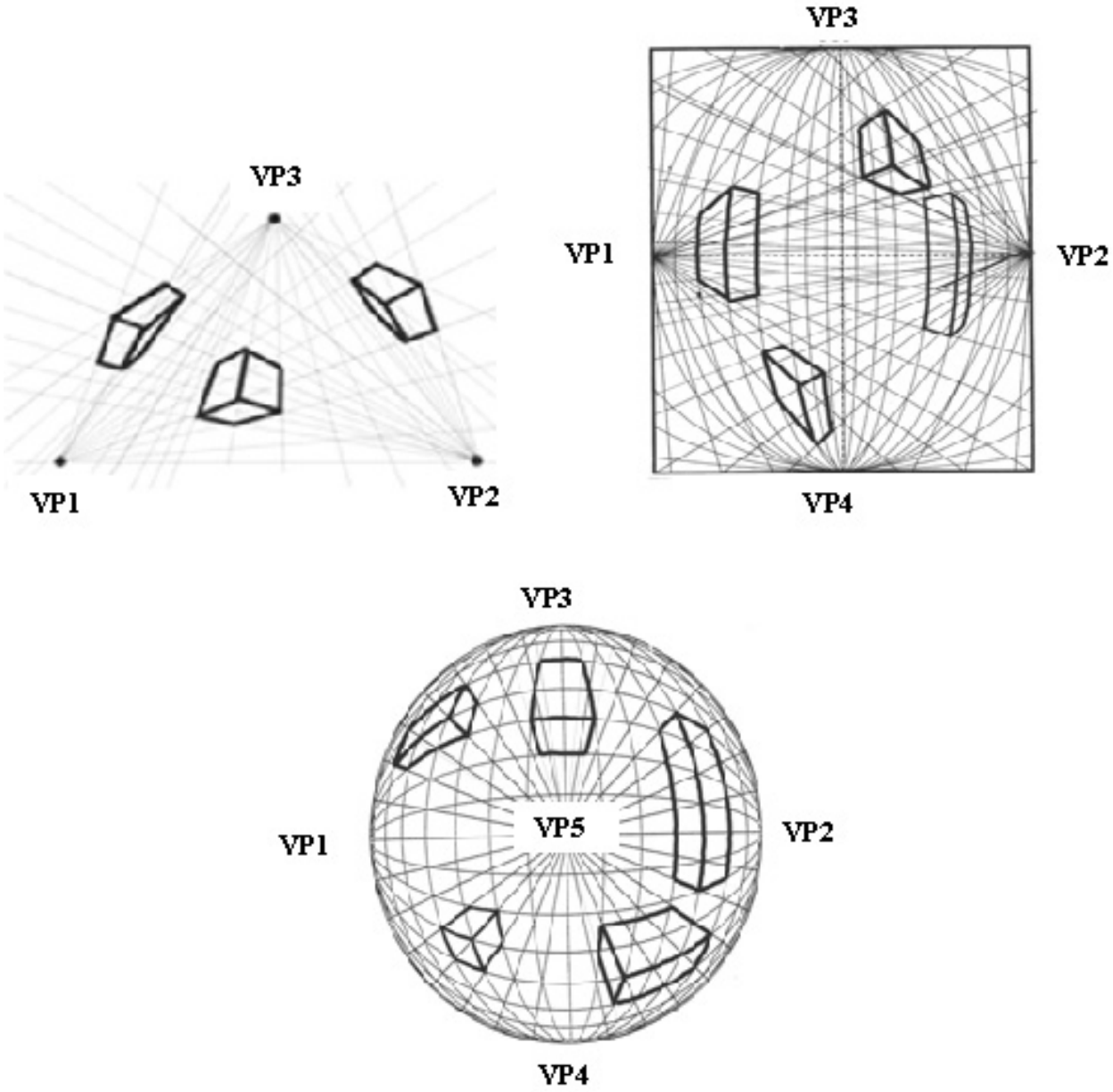


Figure 6.12 Perspective with 3, 4, and 5 vanishing points

The perspectives with 4 and 5 vanishing points, shown in Figure 6.12, were drawn using the meshes available in Dick Termes book.

To illustrate how a perspective with a larger number of vanishing points, say, with four, imagine we are located on the 40$^{th}$ floor of a building, looking at a neighboring 80-floor building. By looking upwards and then downwards we would have the sensation that vertical lines would converge to points above and below us.

The reader may be asking why such notions are presented in this book. The reason is that these perspectives, although not well known, do allow us to create interesting artwork. To try to show how this might be done, imagine an observer located in the center of a spherical surface made of transparent plastic so that everything around is seen through that spherical surface. Now let the observer be an artist painting on the internal surface of the plastic sphere everything that is seen, as illustrated in Figure 6.13.

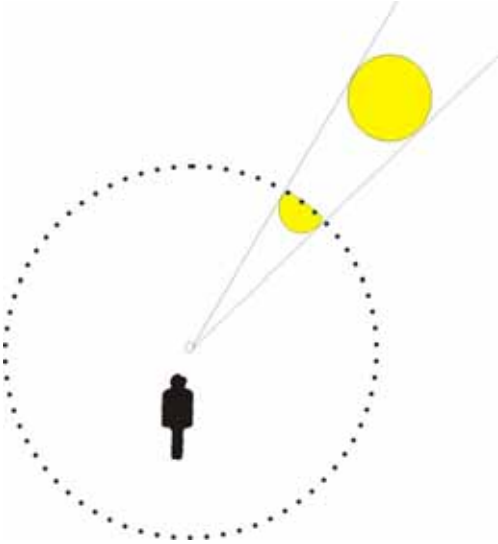

Figure 6.13 Observer at the center of a transparent sphere

After the painting is finished, the plastic sphere is hung from a fixed point and rotated about the vertical by a motor. Looking at it from the outside, the successive images would appear as seen by the observer inside the sphere. Dick Termes used such an idea in his series of spheres known as “Termespheres”. Figure 6.14 shows three views that are sufficient to display his work called “Gargoyles of Saint Denis”, which is based on the famous Cathedral of Saint Denis in Paris; the exotic figures (Gargoyles) are in the outside of the actual Cathedral.

Figure 6.14 “Gargoyles of Saint Denis” - Dick Termes

Another example of a work by Dick Termes is shown in Figure 6.15. It is called “Hagia Sophia” and is based on a Church in Istanbul, Turkey.

Figure 6.15 Hagia Sofia - Dick Termes

Figure 6.16 shows another work by Dick Termes, known as “Brain Strain”, which is a very interesting perspective with a large number of vanishing points.

Figure 6.16 Brain Strain - Dick Termes

Dick Termes work is very interesting and we suggest the reader to research it in more detail.

## 6.5 Linear perspective and color perspective

**A** linear perspective, presented so far, is a technique to represent 3-dimensional space. However, there is another method for creating the sensation of depth using colors: darker colored objects create the sensation of them being in the background, with lighter colors in the foreground. There are no rules for constructing such a type of perspective, and the resulting effect depends entirely on the artist's skill and sensitivity.

## 6.6 Final considerations

After all, was the perspective the final and ideal solution for representing space? It was widely used during the Renaissance, but it did not have the same meaning for some artists hundreds of years later, including Paul Cézanne (1839-1906), who is considered to the father of modern art. Perspectives were invented to create the illusion of depth, and that was not what Cézanne was interested in doing. One can then conclude that it is impossible to establish very definitive rules in a creative and fertile field such as the arts.

Basic notions of perspective were presented in this chapter and several works of art were examined to illustrate its use. However, this is a vast and interesting subject, and we suggest the reader to examine additional artwork and decide on how to proceed when working with perspectives.

# 7

# The golden ratio

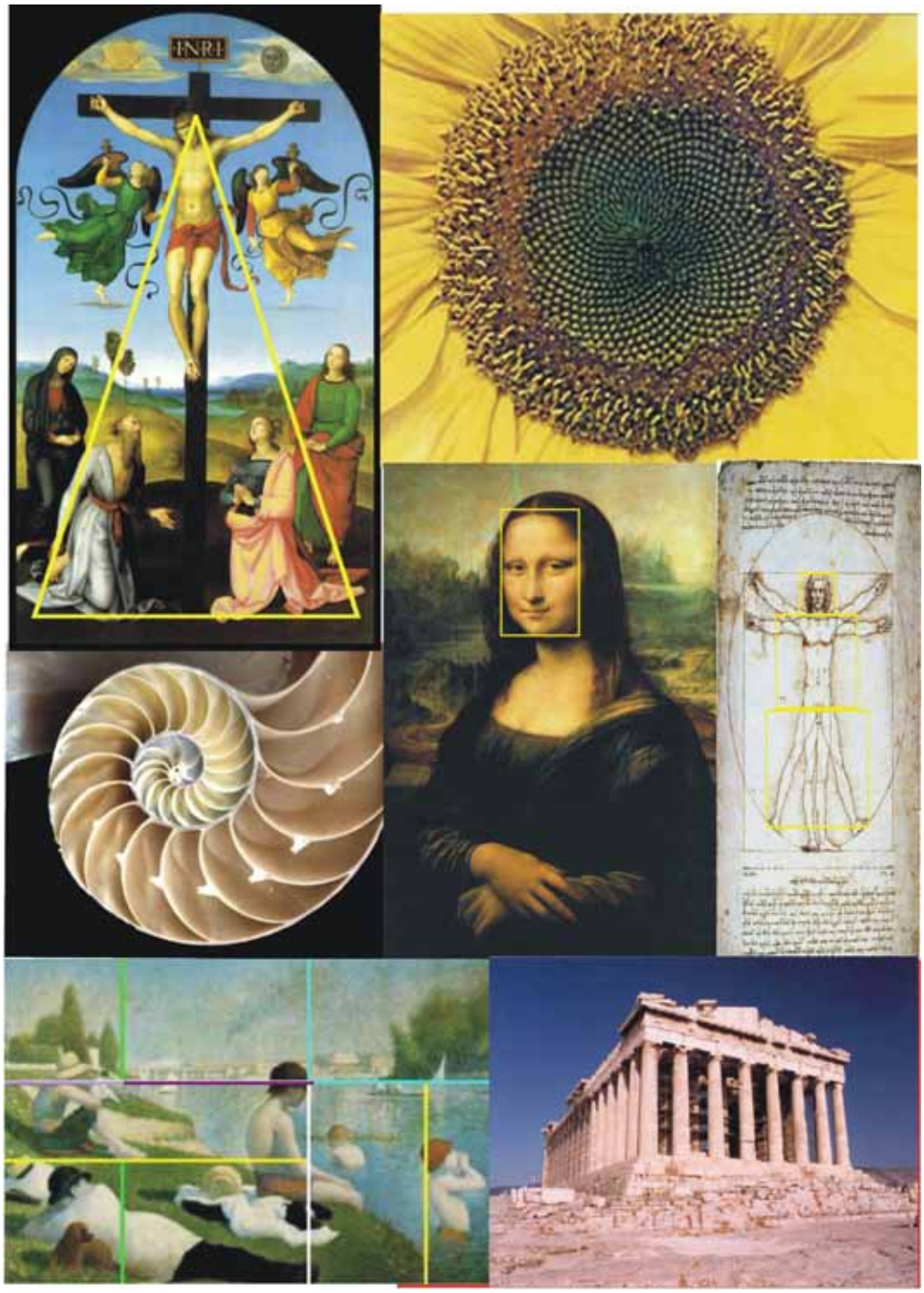


**Figure 7.1 The Golden World**

The golden ratio, also known as golden proportion, divine section, divine proportion, golden mean, has attracted the attention of mathematicians, philosophers, and artists. Its mathematical definition is clear, but its supposedly occurrence in nature and its use in human work, such as in Leonardo Da Vinci's work, is controversial and has been the subject of a number of studies.

### 7.1 Fundamentals of the golden ratio

Figure 7.2 shows a straight line segment divided into two parts: a larger one of length a, in blue, and a smaller one of length b, in red; the total length of the segment is then a+b.

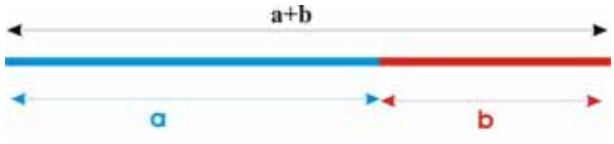


Figure 7.2 Dividing a segment in the golden ratio proportion

The segment is said to be divided in the golden ratio proportion when the ratio of the total length (a+b) to the length of the larger segment (a) is equal to the ratio a/b, i.e., when

$$(a+b)/a = a/b$$

Euclid presented a study of the subject matter in his book "The Elements", and used the expression "division of a segment in mean and extreme ratio". As mentioned earlier, such a ratio is known by several names, including the more recent ones Golden Ratio and Divine Proportion, with the latter appearing during the Renaissance. The name "golden ratio" is appealing and widely used, and is used in this book.

Figure 7.3 shows the ***approximate values*** for the lengths of a 1 meter segment divided in the golden ratio proportion.

Notice that

1.00/0.618=0.618/0.382=1.618

Figure 7.3 Golden ratio example

The ratio of the two line segments shown in Figure 7.3 is 1.618, but let us analyze this value in more detail. To begin with, notice that the lengths of those line segments were referred to as ***approximate values***. In fact, the relationship (a+b)/a = a/b is an equation that can be solved for the ratio a/b (or for b/a), and the solution is a number that is called an irrational number, like the number $\pi$ (pronounced pi). The solution for a/b in (a+b)/a = a/b is commonly denoted by the Greek letter $\phi$ (pronounced fee), and its more exact value is

$$\phi = 1.61803\ 39887\ 49894\ 84820....$$

The reader will find in many articles the value 1.618 for $\phi$, which will also be used here, but note, again, that this is an approximation.

## 7.2 Golden geometric figures

The number ϕ shows up in the dimensions of many geometric figures, which led to its importance and probably to the attention of many mathematicians.

### 7.2.1 Golden rectangles

Any rectangle for which the ratio between the dimensions of its larger and smaller sides is equal to ϕ, as shown in Figure 7.4, is called a golden rectangle.

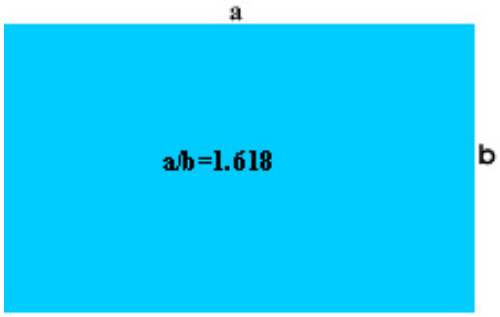


Figure 7.4 Golden rectangle

Several golden rectangles are shown in Figure 7.5 to familiarize the reader with them; their importance is evidenced later when the use of the golden ratio is studied in connection with works of art.

Figure 7.5 Golden rectangles

### 7.2.2 Golden triangles

Figure 7.6 shows two special isosceles triangles with internal angles equal to 72, 72, and 36 degrees, and 36, 36, and 108 degrees, respectively. For these triangles, the ratio a/b of their sides is equal to the golden ratio 1.618. Such triangles are called golden triangles.

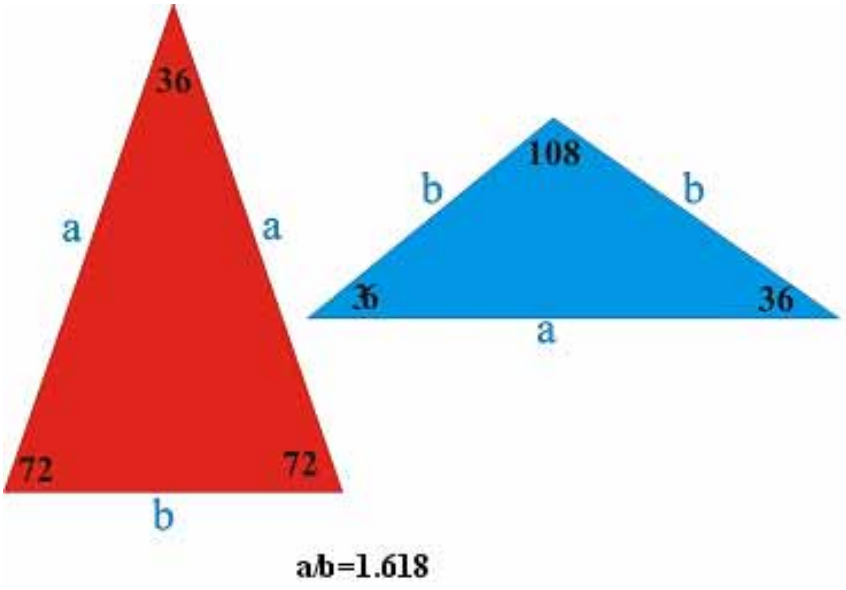


Figure 7.6 Golden isosceles triangles

**7.3 Mathematics of the golden ratio**

The golden ratio is associated with an important numerical sequence, known as Fibonacci sequence. Leonardo Fibonacci, initially known as Fibonacci of Pisa, was born in the Italian city of Pisa around the year 1170. He was well acquainted with the Islamic and Arab mathematical knowledge of that time, and that sparked his mathematical curiosity even further.

Fibonacci wrote several books, and perhaps the best known is his 1202 treaty on algebra and arithmetic entitled "Liber Abaci" (whose common translations are "book of abacus" or "book of calculus"). He addresses several interesting problems in one of the chapters, the best known being the growth of population of rabbits. "A pair of newborn rabbits is placed in a cage in the first day of the year. How many pairs of rabbits will be in the cage on the last day of the year, assuming that rabbits can mate when they are one month old, that each female generates one pair of rabbits each month, and that no rabbits die during the entire year?" Perhaps the solution is not obvious, but by making a list as shown in Table 7.1 it is seen that the number of pairs of rabbits increases each month according to the sequence indicated.

| ***Month/Day*** | ***Pairs of Rabbits*** |
|---|---|
| 1/1 | 1 |
| 2/1 | 1 |
| 3/1 | 2 |
| 4/1 | 3 |
| 5/1 | 5 |
| 6/1 | 8 |
| 7/1 | 13 |
| 8/1 | 21 |
| 9/1 | 34 |
| 10/1 | 55 |
| 11/1 | 89 |
| 12/1 | 144 |
| 12/31 | 233 |

Table 7.1 Number of pairs of rabbits after each month

Thus, after one year the number of pairs of rabbits will be 233.

It can be easily verified that the sequence of numbers in the right column of Table 7.1 has interesting properties. First, it is seen that each of the values in that column is equal to the sum of the previous two values; for example: 2=1+1, ... , 5 = 3+2, ... , 54 =34+21, ... , 233 =144+89, ...
To solve the rabbits' problem we simply stopped at 233. But, what if the sequence is continued, with each successive value being equal to the sum of the previous two values? For example, after 233 we would have 233+144 = 377, 377+233 = 610, and so on. The sequence would then continue indefinitely as follows: . . . , 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, . . .

Such an infinite sequence is called Fibonacci sequence, in honor of Fibonacci who was the first to study it based on the rabbit reproduction problem.

**FIBONACCI SEQUENCE**

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181,.....,10946, 17711, 28657, .....

Readers interested in literature and read Dan Brown's popular book "The Da Vinci Code", perhaps remember that Jacques Saunière, the Louvre Museum curator and director, wrote in code, after being mortally wounded, a bank account number 13, 3, 2, 21, 1, 1, 8, 5, which the smart Langdon and Sophie soon discovered to be numbers from the Fibonacci sequence written out of order.

But let us leave the Louvre aside, with all its splendor and mystery, and return to the Fibonacci sequence, as it has another interesting property. For this, let us look at the ratio of numbers in the sequence to their predecessor. For example,

233 ÷144 = 1.6180555
28657 ÷ 17711 = 1.6180339

It is seen that, as the number in the sequence increases, the ratio of two successive numbers in the sequence approaches the value $\phi$ =1.61803 39887 49894 84820.... With the aid of mathematics, it can be shown that in the limit, as one proceeds indefinitely with the sequence, the value obtained for such a ratio is exactly equal to $\phi$**,** which shows the association between the golden ratio and the Fibonacci sequence.

### 7.4 The golden ratio in works of art and in nature

Let us now look at the most important part of this chapter, which deals with the possibility that the golden ratio has been used in artistic and architectural work, and of its appearance in nature.

#### 7.4.1 The golden ratio in architecture and in the arts

Let us start with the pyramids of Ancient Egypt, such as the pyramids of Cheops, Chefren, and Micherinos. There are many scholarly works analyzing the structural details of these pyramids, including their measurements. Some authors suggest that the dimensions of the Great Pyramid involve the golden ratio. The problem is that not only erosion affects the accuracy of the measurements but also there are conflicting descriptions by the historians. In addition, it is not known whether the Egyptians knew about the golden ratio and, if they did, one cannot be certain they used it even if some measurements suggest such a possibility.

Figure 7.7 shows the Parthenon, built in ancient Greece under the supervision of Phidias (490 BC-430 BC) and on the initiative of the Athenian leader Pericles (490 BC-430 BC). Phidias also created several sculptures for it. Many authors claim there are several golden rectangles in the Parthenon, one of which frames its outside front facade, but a careful analysis by other scholars point out that one cannot be certain.

Figure 7.7 The Parthenon - Acropolis, Athens, Greece

There is a concrete example of the use of the golden ratio in the twentieth century architectural work of Le Corbusier. Le Corbusier was born on October 6, 1887, in La Chaux-de-Fonds, Switzerland, but spent most of his life in France. Charles Edouard Jeanneret adopted the name Le Corbusier in 1923 when his book "Vers une Architecture" was published. In 1950 he published an important work called Modulor, where he presented a measurement system, based on the golden ratio, to be used in architectural compositions.

Let us return to the past and analyze some important works from the Renaissance. The work shown in Figure 7.8 is The Crucifixion, by Raffaello Sanzio (1483-1520), and some suggest that the golden triangle shown in yellow frame the main figures in the painting. Did Raffaello really have that intention?

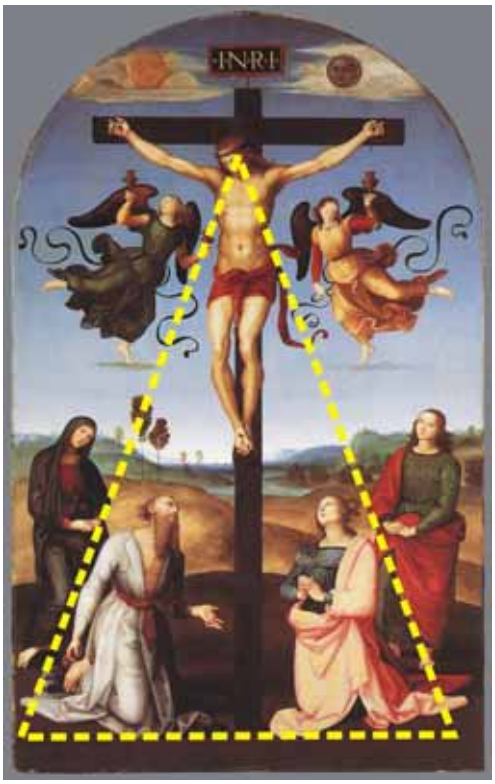


Figure 7.8 The Crucifixion - Raffaello Sanzio - "National Galllery", London, England

The painting shown in Figure 7.9 is the Mona Lisa, by Leonardo Da Vinci. We drew a golden rectangle on the painting, which is a rectangle many suggest was used by Da Vinci to define the dimensions of the face of the Mona Lisa.

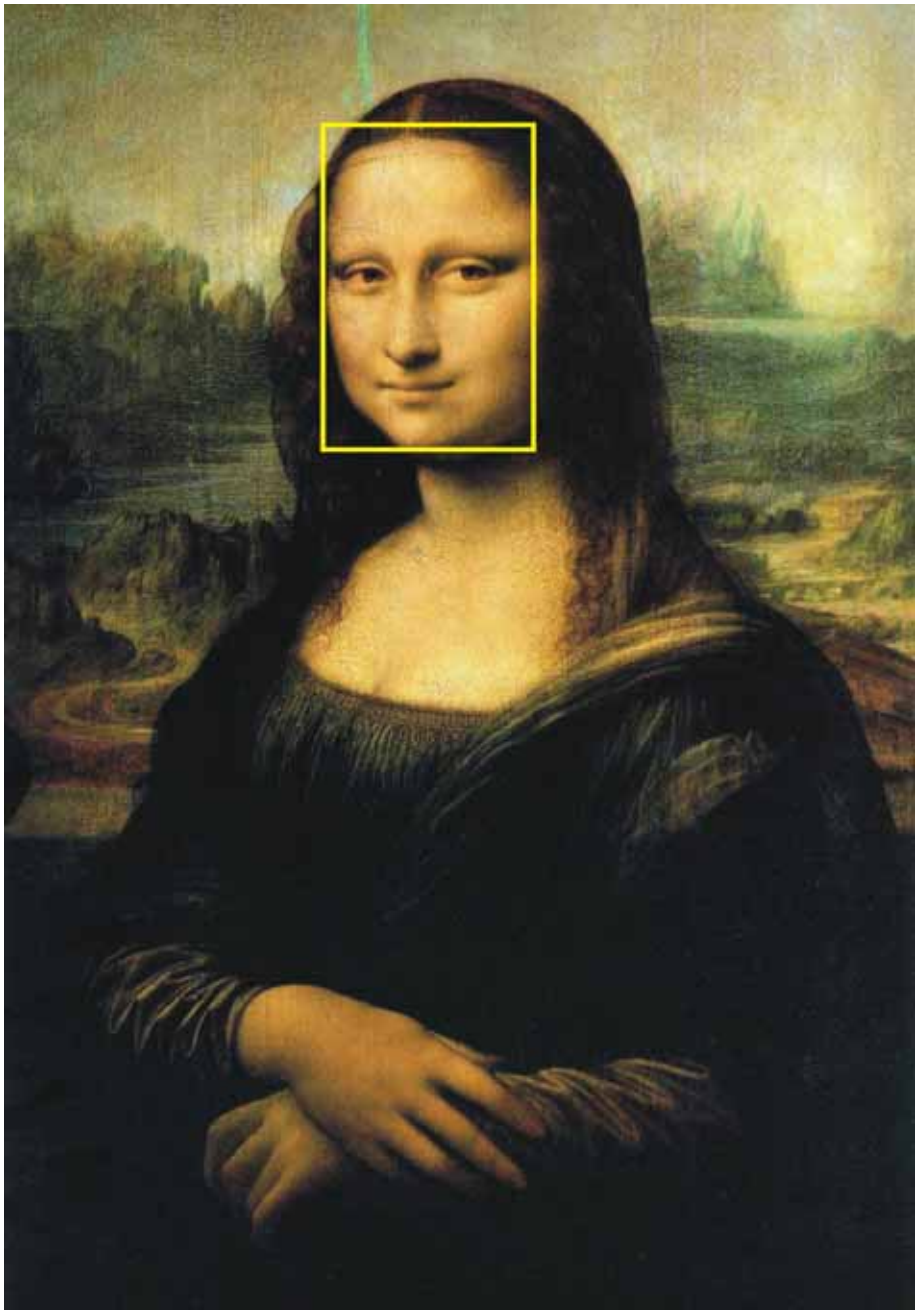

Figure 7.9 The Mona Lisa and the golden ratio - Leonardo Da Vinci- Louvre Museum, Paris, France

Figure 7.10 shows an unfinished work by Da Vinci. It depicts Jerome, one of the fathers of Catholicism and whose translation of the bible, the Vulgata, was published around 400 AC after Theodosius I made Christianity the official religion of the Roman Empire. The painting portrays Jerome with a stone in his right hand, used for sacrificing himself, with a lion in front of him, an animal that has been always associated with him. Many authors allege that Da Vinci drew Jerome in the proportions of the golden rectangle shown in the figure.

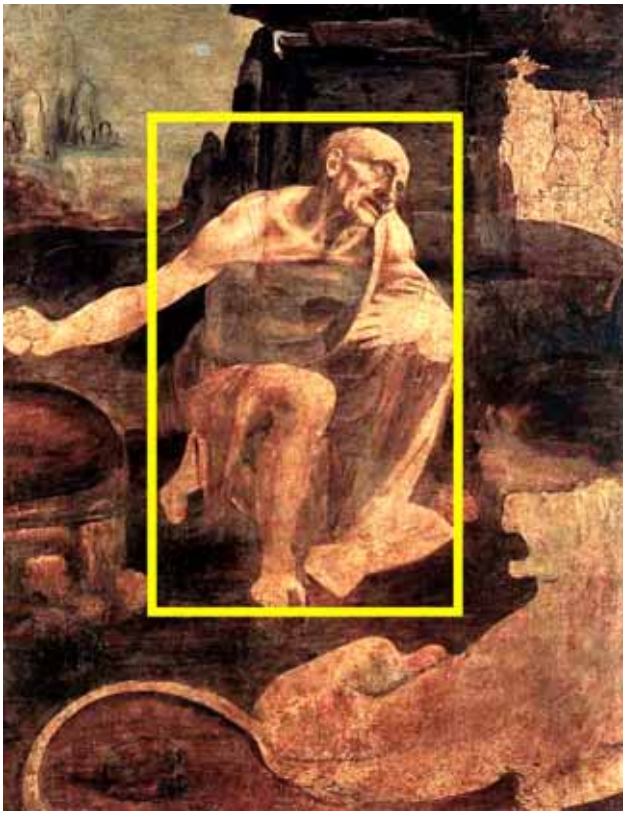

Figure 7.10 Saint Jerome and the golden ratio - Leonardo Da Vinci - Vatican Museum

The allegations mentioned can be subjected to debate. First, with a little practice, several golden rectangles can be drawn framing other important parts of both paintings shown above. In the second painting, the yellow rectangle does not frame half of the right arm, and touches the toes but not the head of Jerome, which defies logic. Again, we are in the realm of speculations, and there is no concrete proof that Da Vinci made use of such rectangles.

An interesting fact pertaining the second painting is that it seems that Da Vinci became acquainted with the Divine Proportion through Luca Pacioli, which happened several years after he worked on that painting without ever finishing it.

Since we are speaking of Da Vinci, let us analyze his drawing known as Vitruvian Man, which is based on a work by Marcus Vitruvius Pollio, a Roman architect and engineer who wrote a 10-volume treatise called "*De Architectura"*, the oldest treatise on the subject matter. Although there are no accurate information on the birth and death of Vitruvius, it may be concluded that such a work was done during the reign of the Roman emperor Augustus (63 BC-14 AD) because Vitruvius dedicated that work to that emperor. That treatise is a masterpiece appreciated, to this day, by architecture lovers because Vitruvius wrote it as a reference for all architectural projects of the expansionist Roman Empire.

In the volume dealing with the dimensions of Temples, Vitruvius argues that the Temples should be constructed based on the dimensions of man since the human body was believed to be a model of perfection. Probably such an idea originated from earlier sources, as Vitruvius mentions that earlier peoples already used well-defined standards of proportion in their creative work.

Vitruvius defined in his studies the proportions between the different parts of the human body, leading to the idea that the arms and legs, when extended, should be bounded by the two most perfect shapes, the circle and the square. This originated what became known as the "Vitruvian Man".

Others tried to use the notions mentioned above, such as Cesare Cesariano (1483-1543) in his 1521 translation into Italian of the work of Vitruvius, and the Italian painter and sculptor Francesco di Giorgio Martini (1439-1501). However, it was Leonardo Da Vinci who came up with a beautiful solution to the

problem around 1492 by using the proportions advanced by Vitruvius and creating his “Vitruvian Man”, probably one of the best known drawings in the world. In the Vitruvian Man drawing, the square and the circle, thought at the time to be two perfect geometric figures, completely frame what was considered to be the perfect man. The yellow rectangles seen in the figure are golden rectangles added by this author to illustrate that it is always possible to enclose parts of the drawing using such rectangles. However, it is obvious that other similar rectangles could have been used, which lead us to be, again, in the realm of speculations.

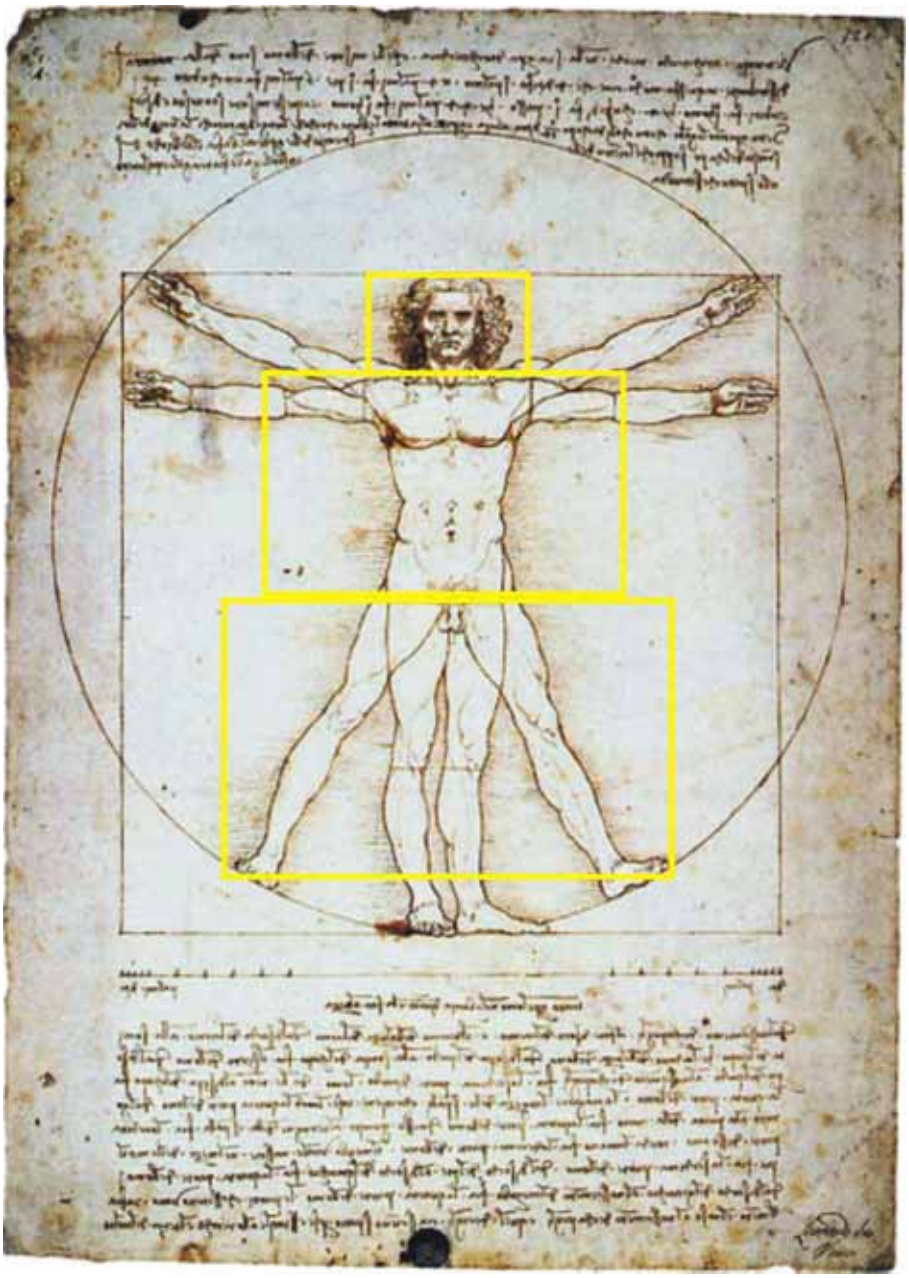

Figure 7.11 Vitruvian man - Leonardo Da Vinci

Skipping through time, it is worth mentioning the work of the neo-impressionist French painter Georges Seurat (1859-1891), who created the so-called Pointillism. Many have suggested that Seurat used the golden ratio to define his work area on the canvas, but there are several studies that show that such was not the case. The same holds for the work of Piet Mondrian.

Let us now look closely at the work of a modern painter, the Surrealist Salvador Dali. In one of his paintings, The Sacrament of the Last Supper, he seems to be inspired by geometric concepts and by their purposely connection with exoteric aspects. The reader can find on the Internet, or in any appropriate book, information about that painting. It is an oil on canvas painting exhibited at the National Gallery of Arts in Washington, D.C., measuring 66 inches by 105 inches. One can immediately notice that the canvas is approximately dimensioned in the golden ratio proportion, as 105 divided by 66 is approximately equal to 1.6 which, for many, is essentially equal to the golden ratio. On the top part of the painting there is a dodecahedron, one of the Platonic Solids, with the figure of Jesus centered in one of its pentagonal faces. Did Dali chose the canvas measurements on purpose?

### 7.4.2 The golden ratio in nature

We saw, up to now, how the golden ratio shows up in several geometric forms, and how several artists may have used it. But what is even more intriguing about the golden ratio is that it is apparently observed in nature, such as in the distribution of leaves along a plant stem, which is the realm of leaf phyllotaxy, and in the distribution of seeds in the flowers of several plants. As an example, a careful look at the sunflower shows that its seeds are distributed along two spirals, one winding to the left and the other to the right, and some suggest that the number of seeds in the spirals varies according to a Fibonacci sequence. But again, such a suggestion is questionable since the quantity of seeds are never the same and a statistical study is unable to establish a relationship between the number of seeds and the golden ratio.

## 7.5 The golden ratio and aesthetics

It is questionable whether or not using the golden ratio yields more aesthetically pleasing forms, as there are conflicting opinions on the subject matter. As seen earlier, it is not possible to affirm with confidence that even famous artists like Da Vinci used the golden ratio in his work. Let us further analyze this interesting topic.

Figure 7.12 shows ten rectangles, only one of which is a golden rectangle, i.e., a rectangle where the ratio of its length to its depth is approximately equal to 1.618. We ask the reader to look at them carefully and decide if any of them is more visually pleasing or calls more attention than the others.

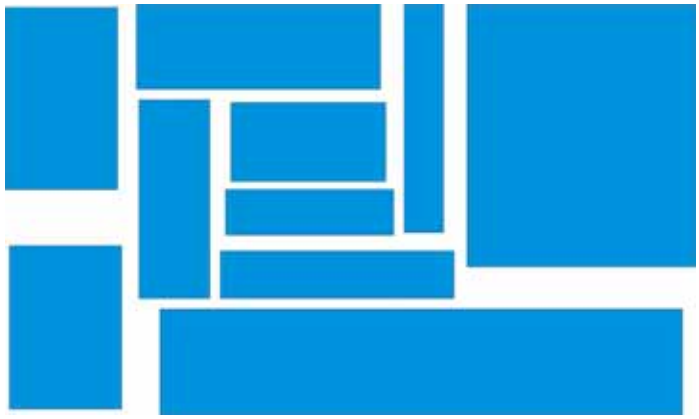

Figure 7.12 Which rectangle is more harmonious?

In reality, only the rectangle at the upper left corner is a golden rectangle. Did the reader find that rectangle to be the more pleasing one by any chance? Did the location of that rectangle in the context of the entire figure affect your observation?

The questions posed above simply indicate that the subject matter is a complex one, and that one cannot easily reach a definitive conclusion. The first investigations to determine whether the golden rectangle is really the most aesthetically pleasing to human beings were done by the German experimental psychologist Gustav Theodor Fechner (1801-1887). In his experiments he presented ten rectangles (which were not the

rectangles shown in Figure 7.12) to a group and asked each one in the group to identify which rectangle was most pleasing. Fletcher concluded that the golden rectangle was the one chosen by the majority. In fact, he went further in his attempt to prove the predominance of the golden ratio by experimenting with hundreds of rectangular objects used by man. However, experiments made by many other researchers were unable to substantiate Fletcher conclusions and, thus, it is not possible to affirm that there is a certain measurement ratio that is more pleasing to humans.

The reader interested in further pursuing the subject matter may want to consult related publications by Mario Livio and other authors.

**7.6 Conclusions**

The golden ratio, a fascinating topic, has drawn the attention of mathematicians and philosophers, mainly due to the assumption that many artists used it in their work. So far, no scientific study has been able to prove that the golden rectangle is the most pleasing to humans, or that there is a measurement ratio with such a property.

# 8

## Symmetry and specular image

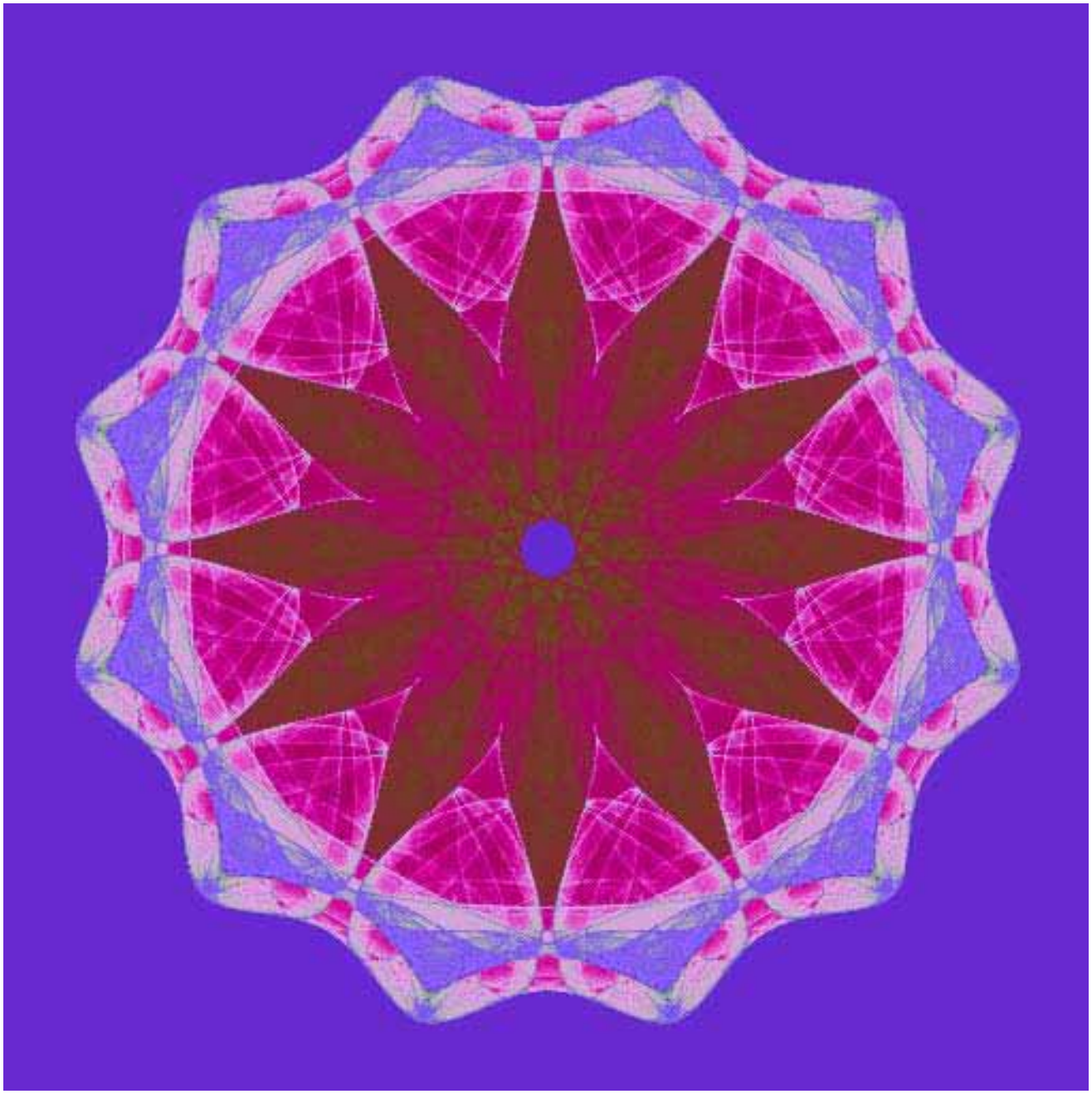

**Figure 8.1 Symmetric image**

**8.1 Symmetry**

Let us first look at the concept of symmetry. By observing Figure 8.2 it is seen that if triangle ABC is rotated 180 degrees about the R-axis, point A will coincide with point A', point B with B', and point C with C'. This happens because line AA' is perpendicular to the rotation axis R, and the distances from A to R and from A' to R are the same, which is a similar situation for points B and B', and C and C', respectively. Figures ABC and A'B'C' are said to be symmetric, and line R is the axis of symmetry.

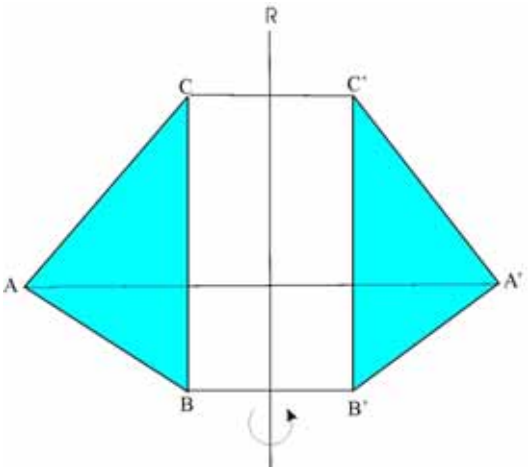


Figure 8.2 Symmetric figures

An isosceles triangle has an axis of symmetry, as shown in Figure 8.3.

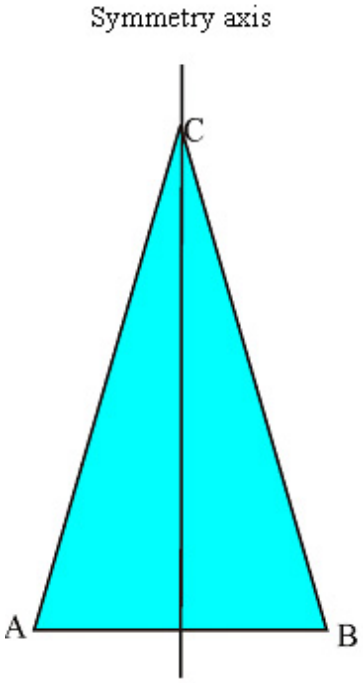


Figure 8.3 Isosceles triangle and its axis of symmetry

A figure may have several axes of symmetry. As shown in Figure 8.4, a hexagon is symmetric with respect to any axis that pass either through two of its opposite vertices or through the middle of any two of its opposite sides.

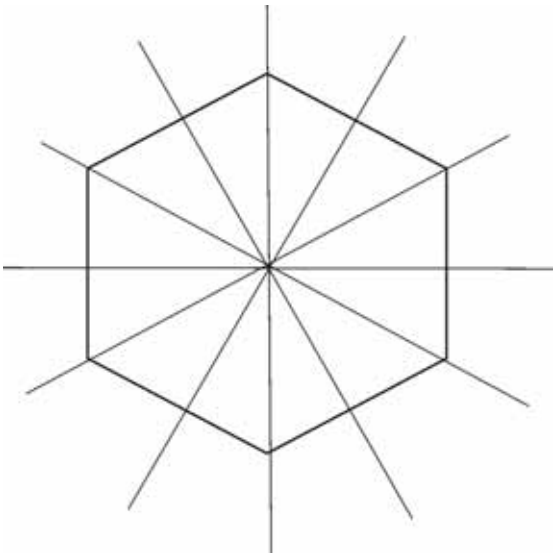

Figure 8.4 Figure with several axes of symmetry

Take a look at the 12-point star shown in Figure 8.5 and observe its symmetry axes, which are drawn in yellow.

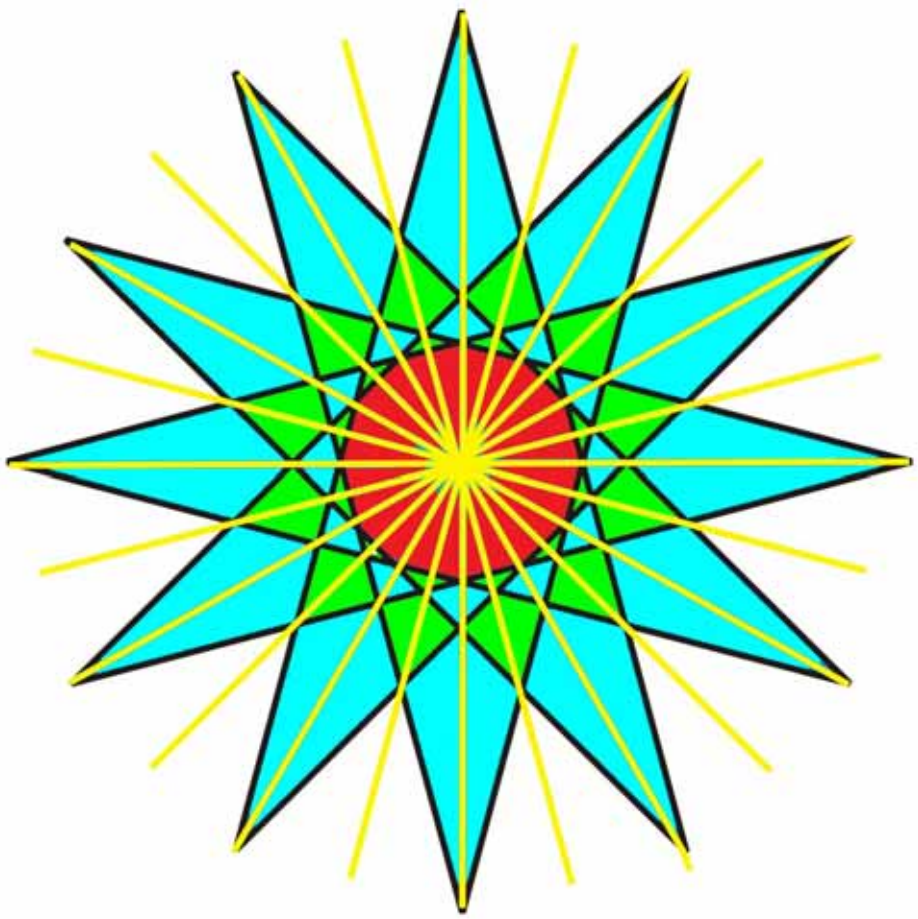

Figure 8.5 Symmetry axes of a 12-point star

**8.2 Specular image**

The image we see when we look at ourselves in a planar mirror is symmetric with respect to the mirror. However, the image is inverted. For example, an earring in our left ear appears in the right ear in the image, and vice versa. Such an image is called a specular image.

Another example of specular image can be seen with a photographic slide (which, incidentally, is no longer popular in photography). Look at any slide and then turn it around and look at its reverse side. What is seen on the reverse side is the specular image of the other side.

Figure 8.6 shows two images of the artwork entitled "Akhenaton". The actual work is shown on the left, and its specular image on the right.

Figure 8.6 Akhenaton - Egyptian Series - Hamello

Another situation where use is made of a specular image occurs when one wants to print an image on a T-shirt, for example, using the so-called "transfer" created with the resources available with a computer. For this, the desired image is first printed on a special surface, called the "transfer", which is then put in contact with the T-shirt and heated with an iron. The problem is that the resulting image on the T-shirt is the specular image of the original figure. For this reason, computer graphic programs normally print a specular image on the "transfer" so that the original image is the one that is correctly transferred to the desired surface, as a T-shirt for example.

Figure 8.7 shows an interesting aspect of the specular image associated with Leonardo Da Vinci's "The Annunciation" painting. In the upper image, which is the original work, the Angel seems to be arriving to "announce" the great miracle to the Virgin Mary. However, the lower image, which is the specular image of the original painting, does not seem to convey such a perception. Apparently, our perception of an image seems to be conditioned to our usual manner of scanning things from left to right, which is intriguing.

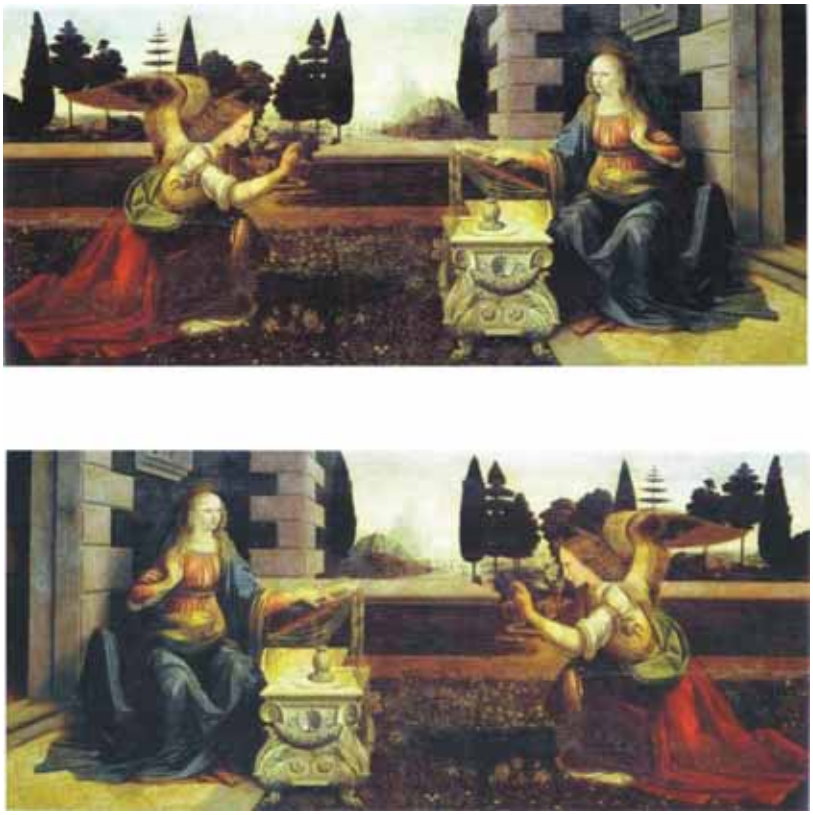

Figure 8.7 The Annunciation - Leonardo Da Vinci - Oil on wood - circa 1472 to 1474 - "Galeria degli Uffizi", Florence, Italy

**8.3 Symmetry in nature**

The precise meaning of symmetry was presented earlier, and symmetric images can be seen in the work of many artists such as M. C. Escher, Barnett Newman, Gare Maxton, to cite a few.

Although mathematically symmetric figures do not appear in nature, there are some plants and animals that are nearly symmetric, exhibiting a high "degree of symmetry". Figure 8.8 shows a Monarch butterfly, which is essentially symmetric relative to its central axis if small imperfections are disregarded.

Figure 8.8 Monarch butterfly

Figure 8.9 shows two orchids as an example of a high degree of symmetry that can be seen in the vegetal kingdom.

Figure 8.9 Examples of symmetry: orchids

# 9

## Geometry and symbolisms

**Figure 9.1 “Big-Bang” - Hamello - Encaustic**

Geometric figures have been associated with symbols since antiquity, probably because their symmetry, beauty, and ever presence in nature, have lead man to closely associate them with something superior, intangible, and perfect.

Symbolic meanings are attributed to the circle and to the sphere for their expressing of totality, perfection, as well as complete and total integration. For some scholars the circle expresses the totality of the psyche in all of its forms. The circular shape is present in manifestations of human feelings such as the ancient worship of the Sun and the Moon, in the Tibetan mandalas, in meditation labyrinths, in the rosaceae seen in stained glass windows in Cathedrals, among others. In the Zen sect, the circle symbolizes human perfection, characterized by enlightenment. The halo in Christ and in Christian Saints has a circular shape. Since symbolism appears in many important artworks, the symbolism associated with some of the geometric forms studied in this book is presented in this chapter.

**9.1 The "Vesica Piscis"**

The "Vesica Piscis" is an important symbol based on the circle. It is generated by the intersection of two identical circles, as shown in Figure 9.2 with one circle centered at point A and the other centered at point B. The area common to both circles, shown in blue, is called "Vesica Piscis" because it resembles a vesicle, i.e., a receptacle. As seen later in this chapter, the "Piscis", which is Latin for "fish", is associated with the symbol adopted by early Christianity.

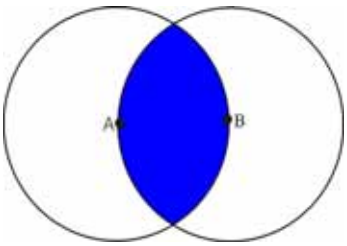


Figure 9.2 The "Vesica Piscis"

Why is such a simple form so important to the extent of deserving special attention? The reason is that both sacred and profane meanings were associated with it in the past. Due to its shape, a mystic interpretation associates the "Vesica Piscis" with the womb of Maria, with Jesus Christ and other Saints often represented inside it, as in the image of a medieval document shown in Figure 9.3.

Figure 9.3 The "Vesica Piscis" womb

Because the two identical circles shown in Figure 9.2 share a common area, the "Vesica Piscis" is also associated with the notion of sharing, in the sense of total and mutual understanding between two Beings; the complete communion between masculine and feminine.

Another interesting aspect of the "Vesica Piscis" is that it originated the fishlike symbol used by the first Christians. Figure 9.4, first obtained by rotating the "Vesica Piscis" shown in Figure 9.2 for easy identification with the earlier Christianity symbol, illustrates the symbol used during the early days of the Roman Empire, before the signing of the Edict of Milan.

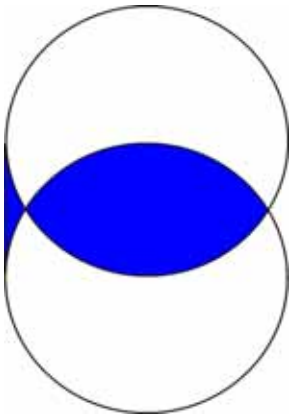

Figure 9.4 The fish: symbol of early Christianity

It is believed that, due to the relentless persecution by the Romans, the Christians held their meetings in secret and used the fish symbol to disclose to other Christians where their meetings would take place. The Edict of Milan, signed by emperors Constantine and Licinius in 313 AD ended that era of terror by proclaiming religious tolerance in the Roman Empire, especially for the Christians. The influence of the "Vesica Piscis" in the design of Gothic churches is also worth mentioning. This is illustrated in the variation of the "Vesica Piscis" shown in Figure 9.5, where the arch pointing towards the sky meant to symbolize the connection with a Higher Being.

We end this brief study of the "Vesica Piscis" by pointing out that it has a connection with the golden ratio discussed in chapter 7; readers interested in pursuing this further can find specific information on the Internet.

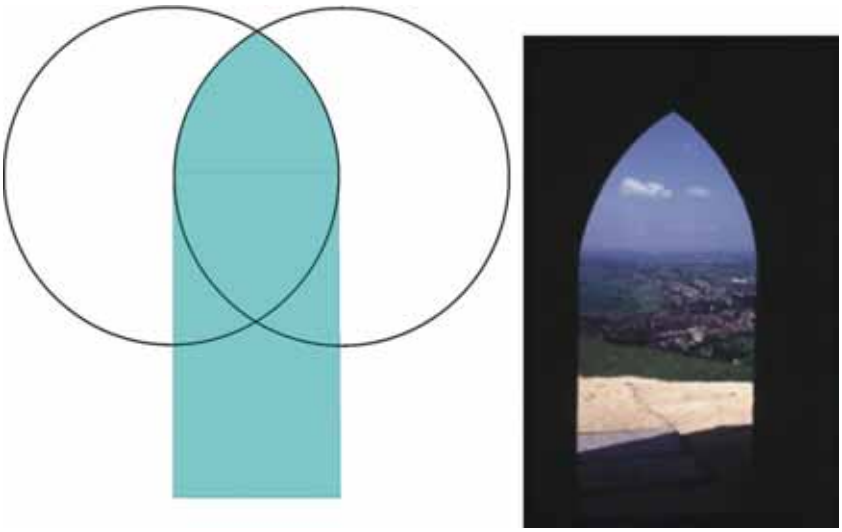

Figure 9.5 Gothic structure based on the "Vesica Piscis"

**9.2 Mandalas**

Mandalas are generally associated with the circle, although they have a meaning that is much deeper than geometry: they are “instruments” used towards self- improvement and self-awareness through meditation. They also have an important role in architecture, and are part of the basic design of many ancient cities, such as Rome. There are several types of mandalas, but our attention will be directed only to the Tibetan mandalas because of their striking beauty and meaning. Tibetan mandalas are built by spreading, with a funnel, colored grains of sand, normally either white or colored powdered marble, on appropriate areas of a planar figure drawn a-priori.

Figures 9.6 and 9.7 show the images of two beautiful Tibetan mandalas, provided by the plastic artist Dar Freeland.

Figure 9.6 Tibetan mandala - Dar Freeland

Figure 9.7 Tibetan mandala - Dar Freeland

Meditation is actually associated with the construction of the mandala. One constructs it when meditating and then destroys it afterwards, with the starting of the construction and the destruction of the finished mandala being accompanied by formal rituals, normally with music and prayers.

Figure 9.8 shows the image of another type of mandala, provided by the English artist Barry Stevens, who has been painting them for many years.

Figure 9.8 “Rainbow spiral” - Barry Stevens

**9.3 Yantras**

Yantras are geometric diagrams that some believe evoke divinity and, thus, use them for meditation. There are many different types of Yantra; the one shown in Figure 9.9 is known as “Shree Yantra”.

Figure 9.9 “Shree Yantra”

The interpenetrating triangles seen in Figure 9.9, some pointing up and some pointing down, are meant to symbolize the union of the masculine and feminine divinities.

**9.4 Labyrinths**

Labyrinths are ancient meandering paths that usually associate a circle and a spiral leading to a center. The purpose is for one to enter the labyrinth and walk towards its center, and then return, while meditating during the entire walk.

Often a labyrinth is thought of as having several confusing pathways where one can get lost and have difficulty in exiting it. Such a structure is a ***maze***, and that is not the case of a labyrinth. Strictly speaking, the term ***labyrinth*** is associated with a non-confusing path, as defined above, used for deep meditation that would lead to a better understanding of the inner self and, upon return to the real world, to a better knowledge of what one wants to accomplish in life.

A maze is a meandering path designed as a test to confuse and to challenge one to find an exit. There is a story in Greek mythology saying that Daedalus, a Greek craftsman, built for King Minos of Crete the "labyrinth of Crete" (actually, a maze) to hold the Minotaur, a creature half man and half bull that devoured human beings. Daedalus revealed the secret of the passageways to Ariadne, daughter of King Minos, and she helped Teseus of Athens to find his way in and out of the maze to kill the Minotaur. That infuriated King Minos, who then imprisoned Daedalus and his son Icarus in the maze. They escaped with wings fashioned with wax, but Icarus flew too close to the Sun causing the wax to melt, leading to Icarus death after he fell into the sea. Although archeologists have not been able to find evidence of the existence of the "labyrinth of Crete", this is, nevertheless, a nice story.

Figure 9.10 shows a sketch of the labyrinth at the Cathedral of Chartres, near Paris, France, built around the year 1200. It is a labyrinth with 11 circuits, i.e., 11 circular paths that get closer and closer to the center, divided into four quadrants. A person walking the labyrinth passes several times through the four quadrants, wondering if the center will ever be reached.

Figure 9.10 Sketch of the Labyrinth at the Chartres Cathedral

The labyrinth shown in Figure 9.10 is very popular and well known, and may be found in stores that sell mystic products.

**9.5 The Star of David, pentagrams, and hexagrams**

The Star of David, the Jewish symbol, consists of two inversely positioned equilateral triangles, with a vertex of one of the triangles pointing upwards and a vertex of the other triangle pointing downwards, as shown in Figure 9.11.

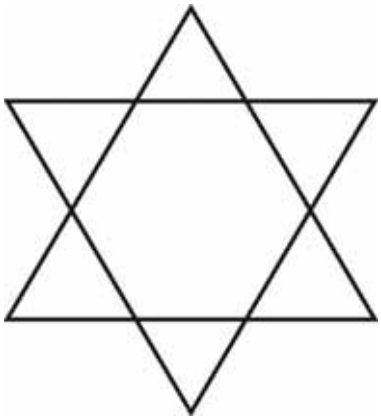

Figure 9.11 Star of David

As seen in Figure 9.11, the two triangles form a six-point star and, for that reason, such a figure is also called a hexagram. According to tradition, the denomination Star of David stems from the fact that such a symbol was used in the shield of King David. It is also known as the Seal of Solomon, as the same symbol was also in King Solomon's ring.

Actually, there is some confusion with these denominations as the Seal of Solomon is sometimes identified with the pentagram instead of the hexagram. Initially, such a symbol was used for decorative purposes, but in 1354 Emperor Charles IV of Prague allowed the Jews to have their own flag with the six-point star at its center. The State of Israel officially adopted the flag, shown in Figure 9.12, in 1948.

Figure 9.12 The Israeli Flag (adopted in 1948)

For its beautiful form, the six-point star based on two triangles lends itself to several interesting geometric variations, two of which, shown in Figure 9.13, may lead the reader to try constructing other forms.

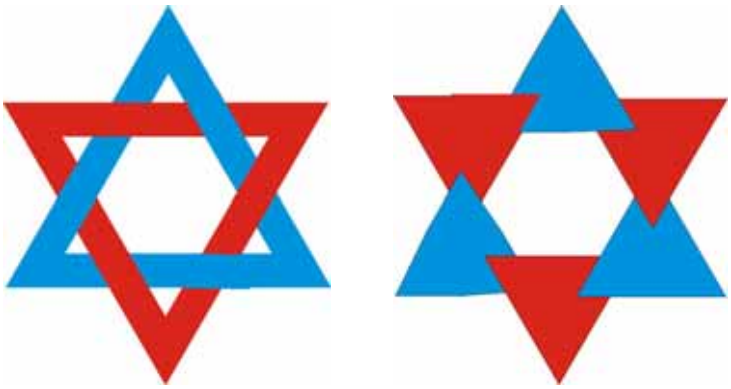

Figure 9.13 Variations in the drawing of the Star of David

The hexagram has been a popular symbol in many cultures, with several religious and pagan associations attributed to it. In one of such associations, the triangles pointing upwards and downwards represent male and female sexuality, respectively, with the two triangles together representing a union. In alchemy, the two triangles separately would represent water and fire and, when together, the union of two opposing forces.

The hexagram inscribed in a circle, shown in Figure 9.14, was also known as trap for the demons, for the wizard standing within that circle could invoke the devil and use it to convert his wishes into reality while being shielded from the devil as long as he stayed inside the circle.

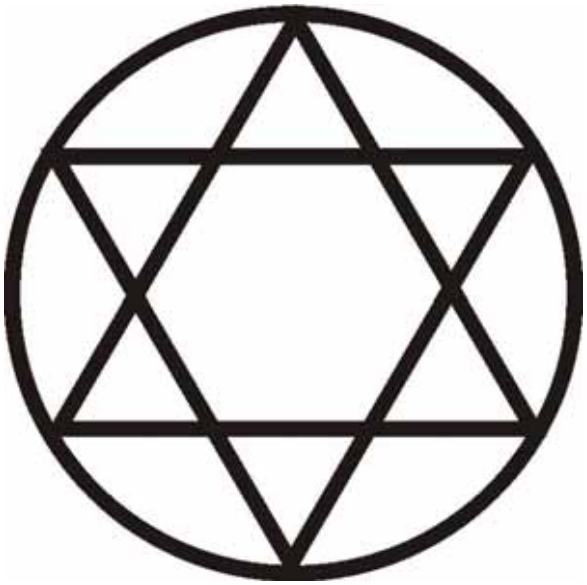

Figure 9.14 Hexagram

The pentagram shown in Figure 9.15 is one of the symbols used in paganism, whose belief in the Sacred Feminine led followers to worship Mother Goddess and the fertility of Gaia, the Mother Earth. The circle around the star represents the divinity to be invoked, and the vertices of the star have the associations shown in the figure. The forces associated with the higher divinity are invoked when the pentagram is drawn pointing up as shown in Figure 9.15, while the brute forces of nature, the inferior divinity, are invoked if it is drawn pointing down.

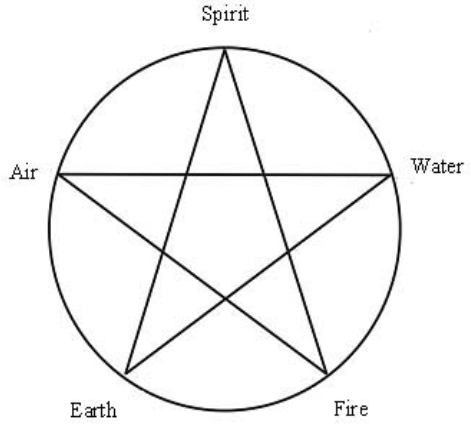


Figure 9.15 Pentagram and the associated elements

**9.6 Spirals and symbolisms**

Spirals are part of many ancient traditions and in particular were widely used in Celtic traditions, the Indo-Germanic people that spread throughout southern Spain, Italy, Britain, the Black Sea, the Balkans, and Asia Minor. Since points on a spiral gradually move away from a central point (or approach a central point) in a cyclic manner, these curves have been associated with time since antiquity.

Figure 9.16 shows the "**Triskell**", the ancient spiritual and social symbol of the Celts, with its three spirals, representing "**love**", "**strength**", and "**knowledge**", sharing a common point of self-knowledge. This is a very interesting symbol, and many artists used variations of it in jewels, sculptures, etc.

Figure 9.16 The "Triskell**"**, a Celtic symbol

**9.7 Additional symbols to be investigated**

Figure 9.17 shows some symbols that are based on geometric forms; readers who are interested in including them in their work may wish to look for specific information on them. Such symbols are associated with ancient religions and are presented here only to spark the reader's curiosity, without any further reference.

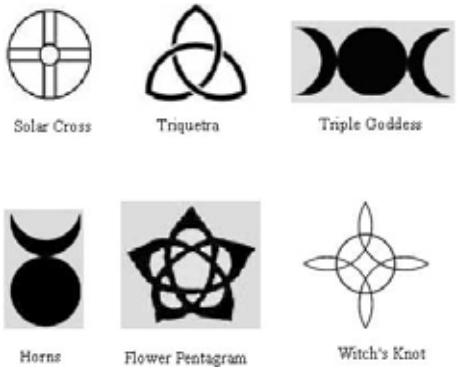


Figure 9.17 Additional symbols to be investigated

**9.8 Polyhedrons and symbolisms: Platonic solids and the elements**

The Greek philosopher Empedocles (492 BC-432 BC) conjectured that everything in nature was composed of four elements: earth, air, fire, and water. Plato, who was born after Empedocles died, conjectured in his famous book Timaeus that the triangle was the basic building block of the universe, and that originated the five solids that were later called Platonic solids. Actually, those solids were known since antiquity, well before Plato. In Timaeus. Plato associates the Platonic solids with the four elements, as illustrated in Figure 9.18.

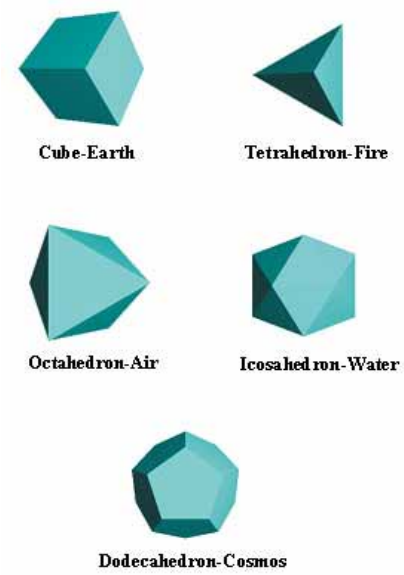


Figure 9.18 Platonic solids and their association with the elements

The cube was associated with the earth for being the most immobile of the solids. The tetrahedron, considered to be most mobile of the solids, was associated with fire, while the icosahedron and the octahedron were associated with water and air, respectively. The fifth solid, the dodecahedron, was associated with the Cosmos, with the stars placed on it by the Gods.

Polyhedrons have always been in wizard and alchemists' "arsenals", and appear in many artworks with some special connotation. For example, look at Durer's work entitled "Melancholia", shown in Figure 3.22. Several scholars interpreted that work by trying to explain the symbolism of each component, including the polyhedron and the sphere. Some have suggested that the polyhedron in that work might represent the Philosopher's stone or the Stone of Saturn.

# 10

## Informatics and the arts

**Figure 10.1 “Primitivo” - Hamello - Acrylic on canvas**

## 10.1 Introduction

The advent and popularity of informatics sparked a radical transformation and cost reduction in the spreading and availability of information throughout the world. Nowadays, we can access any information on the Internet, provided we know how to search for it. We can do things like access banks, view schedule of City events, shop, chat, "visit" museums, simply by using the Internet. Also, many artists create art with the computer using special programs, and many others display their work on the Internet. In this chapter we offer some comments we deem to be of relevance to the subject matter.

## 10.2 Searching information on the Internet

It is an undisputed fact that practically all information one needs may be found on the Internet. In principle this is true, but the problem is how to access specific information that is stored in a network involving many computers all over the world. Given the large amount of stored information available, it is "like looking for a needle in a haystack"! Thus, a mechanism for pointing the way for finding the desired information, such as suggesting "*sites*" where the information sought may be found, is needed.

The idea that revolutionized the Internet originated at Stanford University in 1994, when two Ph.D. candidates, David Filo and Jerry Yang, created a guide for their own use containing "links" that were of personal interest to them. They called it "Jerry's Guide to the Word Wide Web", but changed its name to "Yahoo!" after thousands of people started to use it. In 1995 "Yahoo! Inc." was founded, and it is now a leading worldwide Internet Company.

Several Internet "search engines", as they are called, were created after Yahoo's success. They include Lychos, Altavista, "Cadê" in Brazil (later acquired by Yahoo), and Google, which is currently the most powerful and popular. They continuously search the Internet analyzing the contents of each site, choosing keywords that are then added to a huge database, together with an indication of where those keywords are located.

To search for information on a desired topic, one simply chooses some keywords deemed to be relevant to the desired topic and the database then provides the links to the sites that have the information being sought.

The vast majority of artists, museums, and cultural centers, have their own sites where one can view specific work or even go on a guided "tour". Links to a number of sites visited by this author are listed in chapter 12.

## 10.3 Geometry software

As mentioned in some parts of this book, there are computer programs ("software") for working with geometric forms. Adobe **Photoshop**, by Adobe Systems Inc., and **Corel Draw**, by the Corel Corporation, are two classic programs for the graphic arts and for working with geometric forms.

There are programs that are specifically appropriate to architecture and to engineering, such as **Autocad**, by Autodesk Inc., which is an excellent program that is extensively used for design. There are also programs for mathematics, such as **Mathematica**, by Wolfram Resarch Inc., among others.

For geometry, specifically, there are some programs that are very extensive and complex, and simpler ones for beginners.

This author used three programs in the preparation of this book, namely, Great Stella and Povray for generating three-dimensional figures, and CorelDraw for generating most of the planar figures.

To the reader who is interested in additional programs, we suggest searching the Internet for: Hedron, Kaleido, Kali, Poly, QuasiTiler, QuiltMaker, Ruler and Compass, Sketchpad , SymmeToy, and Tess.

# 11

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| CONTRIBUTING ARTISTS | LINKS |
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| Dick Termes - Termesphere | http://www.termespheres.com/termes.html |
| Russel Towle | http://home.inreach.com/rtowle/Towle.html |
| Ascânio MMM | http://www.ascaniommm.com/ |
| Philippe Hurbain - Legos | http://www.philohome.com/ |
| João Galvão | http://www.galvao.art.br/criticas.php |
| Tom Lechner | http://www.tomlechner.com/ |
| George Hart | http://www.georgehart.com/ |
| | |
| **MISCELLANEOUS** | **LINKS** |
| Polyhedrons | http://www.math.dartmouth.edu/~matc/math5.geometry/ |
| M. C. Escher | http://www.mcescher.com/ |
| Roger Penrose's Tessellations | http://en.wikipedia.org/wiki/Penrose_tiling |
| Jim Mc Neill Polyhedrons | http://www.orchidpalms.com/polyhedra/ |
| Artnet | http://www.artnet.com/ |
| Geometry Center | http://www.scienceu.com/geometry/ |
| Jill Britton | http://britton.disted.camosun.bc.ca/home.htm |
| The Geometry Junkyard | http://www.ics.uci.edu/~eppstein/junkyard/ |
| Western Symbols and Ideograms | http://www.symbols.com/ |
| Sacred Geometry | http://www.geometrycode.com/sg/index.shtml |
| Heesch Tilings | http://math.uttyler.edu/cmann/math/heesch/heesch.htm |
| Geometry in Art and Architecture | http://www.dartmouth.edu/~matc/math5.geometry/unit1/INTRO.html |
| | |
| **ENCYCLOPEDIAS** | **LINKS** |
| Wikimedia - Encyclopedia | http://commons.wikimedia.org/wiki/Main_Page |
| ArtCyclopedia | http://www.artcyclopedia.com/general/alphabetic.html |
| Encyclopedia Britannica | http://www.britannica.com/ |
| "Enciclopédia Itaú de Artes Visuais" | http://www.itaucultural.org.br/aplicexternas/enciclopedia_ic/index.cfm |
| "Barsa" | http://www.barsa.com.br/ |
| | |
| **CULTURAL CENTERS/GALLERIES** | **LINKS** |
| "Dan Galeria" | http://www.dangaleria.com.br/ |
| "Galeria Seta" | http://www.galeriaseta.com.br/principal.php |
| | |
| **MUSEUMS** | **LINKS** |
| Marmottan Monet - Paris - France | http://www.marmottan.com/ |
| Louvre Museum - France | http://www.louvre.fr/ |
| Paul Cézanne's Virtual Museum | http://www.expo-cezanne.com/ |

| | |
|---|---|
| Vatican Museums | http://mv.vatican.va/3_EN/pages/MV_Home.html |
| Russian Museum | http://www.rusmuseum.ru/eng/home/ |
| “Museu de Arte Moderna” | http://www.moma.org/ |
| British Museum | http://www.britishmuseum.org/default.aspx |
| “Museu de Arte Brasileira - FAAP” | http://www.faap.br/museu/ |
| “Museu de Arte Contemporânea da Universidade de São Paulo” | http://www.mac.usp.br/mac/ |
| | |
| **GEOMETRY SOFTWARE** | **LINKS** |
| Povray-Persistence of Vision Raytracer | http://www.povray.org/ |
| Bob Penrose Tiling Generator | http://www.stephencollins.net/Penrose/Default.aspx |
| Great Stella Polyhedron Navigator | http://www.software3d.com/Stella.php |
| Tess | http://www.peda.com/tess/ |